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THE MYSTERY OF
MATTER AND ENERGY

RECENT PROGRESS AS TO THE
STRUCTURE OF MATTER

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R. T. Birge

Dec. 1920

THE ATOM

BY

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TO
MY MOTHER

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Preface

WHILE the present volume is in the main devoted to an exposition of a new theory of the atom, the results which are incident to this theory have assumed an unexpected importance. New theoretical values for Rydberg's constant and Planck's constant have been obtained, from which numerical values of all of the important constants connected with the electrons have been derived, namely Planck's constant, the electronic charge, the masses of the electron and hydrogen nucleus. These agree within the limits of error with the direct experimental determination of these constants.

These new theoretical expressions have supplied the missing equation by means of which the dimensions of the two aetherial constants, specific inductive capacity and magnetic permeability, become separately known, a matter the importance of which has recently been emphasized by Sir Oliver Lodge. Not only have the dimensions of the two aetherial constants been found in terms of length and of time, but those of ordinary mass as well, so that the dimensions of all kinds of quantities, — electrical, magnetic and mechanical, — are capable of expression in terms of length and of time alone. A table of dimensions has been prepared giving the dimensions of the more common units in terms of length and of time, referred to as the space-time system of units. New units of length and of time are considered in place of the centimeter and the second, as a result of which the im-

portance of expressing the specific inductive capacity or magnetic permeability in all electromagnetic equations is very apparent. The velocity of light with the new units becomes numerically unity, and so does twice the Rydberg constant. The specific inductive capacity becomes numerically equal to 3×10^{10}, the velocity of light on the C.G.S. system of units, and to omit to express it in all equations is evidently absurd. Had we always been accustomed to the new units instead of the centimeter and second, there would have been the same natural tendency to omit to express the velocity of light and twice the Rydberg constant as there is now to omit specific inductive capacity.

It is difficult to escape the conclusion that we are one step nearer to a more complete understanding of the real connection between matter and the aether of space, that is to say, an understanding of the properties of the aether itself.

It is not feasible to present this subject without a limited use of mathematical symbols, which were purposely avoided in my former book, "The Mystery of Matter and Energy." The chief purpose in view in that work was a statement of the aims and purposes that constitute a definition of the goal. Since its publication in 1917 much substantial progress has been made toward the attainment of the goal, and it is not now necessary to change the views expressed therein. The mathematical sections of this work are, however, of the simplest kind which students who have followed the common undergraduate courses in the colleges may read. Much may be obtained from the work without following the mathematical processes at all.

June 14, 1919.

NOTE

Since writing the above preface the attention of the scientific world has been focused upon the recently announced results obtained during the total eclipse of the sun, May 29 of this year, a report of which has just been made to The Royal Society of London. These announced results support the so-called "Relativity Theory of Gravitation" due to Professor Einstein. Lest there may some confusion exist in the minds of those not familiar with Professor Einstein's theory because of the name which has been applied to it, some remarks upon this subject seem to be required, because the subject of gravitation is discussed within these pages.

In a report to The Physical Society of London on the "Relativity Theory of Gravitation," Professor A. S. Eddington has summed up the matter on the last page (91) in the following words, "In this discussion of the law of gravitation, we have not sought, and we have not reached, any ultimate explanation of its cause. A certain connection between the gravitational field and the measurement of space has been postulated, but this throws light rather on the nature of our measurements than on gravitation itself. The relativity theory is indifferent to hypotheses as to the nature of gravitation, just as it is indifferent to hypotheses as to matter and light."

The recent result from the eclipse may be regarded as one of the first, if not the first, experimental proof of the truth of the theory of relativity, but it seems to the author to be a misnomer to call the Einstein theory a theory of gravitation, because it deals with one phase only of a much larger general theory, which must assign a cause for the gravitational force. The Einstein theory

admittedly assigns no cause for the force, and does not connect it with the atoms of matter or with the motion of the electrons within these atoms. The theory developed within these pages does connect the gravitational force directly with the motion of the electrons within the atoms. To obtain these results it is pointed out that the theory of relativity is required, for this theory is involved in the recent modification of electromagnetic theory due to Mega Nad Saha, who makes use of the four-dimensional space of Minkowski and the relativity theory. The common form of electromagnetic equations is required to be modified to obtain the results described in these pages, and the establishment of the relativity theory strengthens the argument for the author's theory of gravitation.

There is no conflict between the theory of Professor Einstein and that here given. On the contrary, they supplement each other, both depending upon the theory of relativity. The reason for adding these remarks is the thought that it may naturally occur to any one that there is room for but one theory of gravitation. So there is, but such a theory must go to the root of the matter and assign a cause for the gravitational force, and the Einstein theory does not claim to do this, but deals with a single phase of a broader comprehensive theory. It is hoped that this explanation will have the desired effect of removing any misunderstanding because of a supposed conflict of theories.

The delay in the publication of this work because of a printer's strike has afforded an opportunity to refer to these very recent results obtained from the eclipse of May 29 before the book goes to press.

Albert C. Crehore

November 19, 1919.

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THE ATOM

I

THE question how best to present the new theories contained in this volume has given the author some anxious moments. It has been customary among physicists to publish papers representing original contributions to physics in the best technical journals and the Proceedings and Transactions of the Learned Societies, and the author has followed this custom for many years. In the present instance, however, the space required for a proper presentation of the subject is so great that it would of necessity extend the publication over a period of several months in a series of articles in the standard journals. The alternative course of condensing the work into less space for this purpose has been considered and rejected; for it is believed that in this instance the force of the argument would be weakened, not strengthened, by abbreviation. It is not contended that this is true in general, but it seems to be particularly true of the present account. There are many phases to the questions dealt with, and all of them play a very definite part in assisting any one to form a comprehensive idea of and competent judgment of the whole.

It would be false modesty to try to disguise the fact that the subject of atomic theory is treated in a new form from the beginning to the end of the volume, and I

have endeavored to make clear the reasons why a new treatment seems imperative. This has made it necessary to emphasize the points where the current theory of the atom, that due to Dr. N. Bohr, is deficient. And this is the more necessary because this theory has made a very strong appeal to physicists, who with some reservations may be said to have adopted it as their guiding theory. To do this will be regarded by those who have their faces set only towards progress as no disparagement of the work of Dr. Bohr. Indeed, the author takes this opportunity to say that he regards the work of Bohr as most suggestive, and as marking an epoch in the progress of atomic theory.

It will not be beside the point to state briefly the principal objections to the Bohr theory. It is held by this theory that a uniform and constant frequency of vibration of something takes place while an electron is changing over from one circular orbit to another circular orbit of smaller radius about the nucleus of the atom. What it is that vibrates has never been shown and cannot even be imagined. The path of the electron in changing over cannot possibly be a simple circular path, and it is impossible to attribute the uniform vibration supposed to exist to the form of the path of the electron.

The new theory leaves us in no doubt on this matter, and attributes the vibrations emitted during the radiation from a gas directly to the forms of the paths followed by the electrons themselves in returning to their normal orbit after displacement.

This view of the matter does not require an infinite number of possible stable orbits which are postulated in the Bohr theory, but, in hydrogen at least, assumes that there is but one fixed orbit in the normal state when not

radiating energy. When the electrons receive energy from without, one of them is driven out to some maximum distance from the nucleus, depending upon the amount of energy received, and returns immediately thereafter to its former normal orbit, provided no second pulse of energy is received in the meantime. The radiation of the energy, which has been received, takes place as it is returning to the original orbit, and the paths followed by the electrons are responsible for the frequencies of the vibrations observed in the spectra.

It has been attempted to work out the forms of these paths that are possible paths such as will emit only the radiation frequencies in the observed spectra, and, as a guide for this purpose, use is made of a theorem which has been established by means of the current form of electromagnetic theory. Reference to this can scarcely be made in these introductory remarks.

The physical processes attributed to the electrons are, therefore, very different in the Bohr theory and in the author's theory. In the former one operation of an electron in changing over from one orbit to another is supposed to emit but one single vibration frequency, while in the latter an infinite series of frequencies is emitted by the electrons in one operation, that of returning toward the nucleus again after one excursion.

By applying the principle of the conservation of energy, and by the use of the Einstein equation, which makes the energy radiated equal to Planck's constant times the frequency of vibration, it has been possible to obtain the total energy radiated from the system by summing up the terms of an infinite series, since an infinite series of frequencies is emitted in one return of the electrons from a single excursion. A simple expression for the energy required to separate the electrons from the nucleus, if

Where does single-line spectra come into the

this energy happens to be received while the electron is at its maximum distance, has been obtained. From this energy the ionizing voltages required for hydrogen have been obtained, and they have been compared with the recent experimental observations on hydrogen. The agreement between these theoretical and experimental values is remarkable and affords strong support for the theory. The minimum ionizing voltage obtained from the theory is 11.132 volts, and the maximum 15.5 volts. Experimentally nothing is observed to happen in hydrogen until 11 volts is passed, and a recent experimental result has discovered a new type of ionization in hydrogen at about 15.8 volts, which is in agreement with the upper limit given by the theory. The Bohr theory gives no indication of any ionizing voltage above 13.54 volts, which is too low. Reference must be made to the text for a table of the other ionizing voltages between these two outside figures.

The new theory leads to the belief that the Rydberg constant, which occurs in the formulae of the spectra of every element whose spectra have as yet been reduced to formulae, is connected very closely with the properties of the atomic nucleus. A new expression for the Rydberg constant, different from and simpler than that given by Bohr, has been obtained from the well-known Lorentz mass formula for an electrical charge at slow velocities,

$$m = \frac{4}{5}\frac{e^2}{c^2a}.$$

2/3 I thought

The new value for the Rydberg constant, K, is

$$2K = m_H\left(\frac{c}{e}\right)^2.$$

c = vel. light

If we solve the Lorentz mass formula for the radius,

$2K = m_H\left(\frac{c}{e}\right)$ $m_H = 1.6626 \times 10^{-24}$ using 96,5?0.4 $+4.774 \times 10$

K = 3.2815

correct is 3.28982 .2 % off

$c = 2.9986 \times 10$

a, and apply it to the nucleus of the hydrogen atom having a mass m_H, we have

$$a = \frac{16}{5}\frac{e^2}{c^2 m_H},$$

in which it is seen that the reciprocal of the expression $m_H\left(\frac{c}{e}\right)^2$ involved in the Rydberg constant occurs.

Numerical values of the charge on the single electron, the mass of the hydrogen atom, and the mass of the electron have been obtained simply by the use of this expression for the Rydberg constant coupled with the well-known expression for the Faraday constant, involving the electrochemical equivalent of an element in electrolysis, and the Bucherer constant ratio of e to cm_0 as follows,

e =	4.763×10^{-10}	electrostatic units,
m_H =	1.658×10^{-24}	grams,
m_0 =	$.898 \times 10^{-27}$	grams.

Try again with best val

The only three experimental constants which have entered into the determination of these two values are the Rydberg constant, poor 3.290×10^{15}, the Faraday constant, poor 9649.4, and the Bucherer constant, 1.767×10^7, each of which constants are known with exceptional accuracy. These results obtained from the new formula for the Rydberg constant above given are within the limits of accuracy of the independent experimental determination of e, m_H and of m_0, and there are strong reasons for believing that the expression for the Rydberg constant is a true relation between the quantities involved, and that these are the correct values of e, m_H, and m_0. The values obtained by Millikan are $e = 4.774 \times 10^{-10}$; $m_H = 1.662 \times 10^{-24}$.

? ? poor 1.77 be

$$K = \frac{c \times 1.008}{2e \times \text{Faraday}} = 3.2815$$

$\times 2.9986$ 4.774 9647.04

If K = 3.28982 then gives e = 4.7634 .22% off

Bohr made it come out that way

?

The one thing that has held physicists to the Bohr theory and has helped the theory more than anything else, is the fact that a theoretical expression has been obtained for the Rydberg constant as follows,

$$K = \frac{2\pi^2 m_0 e^4}{h^3}.$$

When numerical values of these constants are substituted in this expression, the agreement with the Rydberg constant is surprisingly close to the third significant figure. There seems to be no support in the new theory for this expression for the Rydberg constant, and it is believed that the expression does not represent a true physical equation. The reasons for holding this view will presently be given, but first it seems worth while to point out that there exist two other numerical coincidences in this expression which are just as close as the agreement with the Rydberg constant. The significant figures in the Rydberg constant are very close indeed to the significant figures in $\pi^2/3 = 3.289,868$. The usual value of the Rydberg constant is 3.290×10^{15}, 3.28982 which differs by only 1.3 in the fifth significant figure. It must be, therefore, that the quantity $6m_0e^4/h^3$ is very close indeed to 10^{15}, an even number, for the Rydberg constant may be written

$$K = \frac{\pi^2}{3}\frac{6m_0e^4}{h^3}.$$

Substituting the values of the constants as given by Millikan in the second factor, we find that it equals $0.999,53 \times 10^{15}$, a value very close to 10^{15}.

It seems to the author to be very unfortunate that the Lorentz form of equations in electromagnetic theory has been developed from fundamental equations that

omit to express the specific inductive capacity of the medium, and by implication at least leaves one to understand that it is of little consequence as being unity in value and devoid of dimensions in terms of length and time. That this is the common practice there is little doubt. A reference to the work of Schott on Electromagnetic Radiation, which may be regarded as a representative treatise on this subject, shows that the specific inductive capacity, k, is ordinarily not expressed in any of the fundamental equations of electromagnetic theory, and indeed if there is any reference to it in this work it has escaped the author's search. This electromagnetic theory has led to expressions for quantities that are equated to each other, and yet their dimensions are different unless we are prepared to admit that k, the specific inductive capacity, is dimensionless in terms of length and time. The Lorentz mass formula above given may be cited as a first example of the meaning. In the form expressing the radius of the atomic nucleus above, the dimensions of the left side of the equation are simply those of length, L. On the electrostatic system of units the dimensions of quantities are expressed in terms of the four fundamental quantities L, M, T and k, and the dimensions of the expression on the right of the equation are Lk^{-1}, and not simply L, as in the left member. If k is dimensionless in terms of L and T, then it becomes a true physical equation, but if it is not, then it leads to erroneous results to write these equations without the k. It is believed that the true dimensions of k, the specific inductive capacity, are those of the reciprocal of a velocity, namely $L^{-1}T$, and we are not prepared to admit that k is dimensionless in terms of L and T. If this is so, then the k must always be expressed and not omitted in order

to make true physical equations, and we will have to write the Lorentz mass formula

$$ak = \frac{4}{5}\frac{e^2}{c^2m}.$$

Now, the reciprocal of the quantities on the right side of this equation occur in the expression for the Rydberg constant given above. The dimensions of the Rydberg constant are simply those of a frequency, or the reciprocal of a time, namely T^{-1}. Hence, the dimensions of the quantities on the right of the above equation should be those of the reciprocal of the Rydberg constant, namely simply a time, T. By giving to k the dimensions of the reciprocal of a velocity, $L^{-1}T$, the left member, ak, does become simply a time, T, thus agreeing with the Rydberg constant. On the electrostatic system of units the dimensions of the expression we have given above for the Rydberg constant, namely

$$2K = m_H\left(\frac{c}{e}\right)^2,$$

are $L^{-1}k^{-1}$. If, now, k has the dimensions of the reciprocal of a velocity, $L^{-1}T$, then these dimensions are simply T^{-1}, and the expression represents the Rydberg constant not only in magnitude but in dimensions.

Now the equation connecting k and μ, the specific inductive capacity and the magnetic permeability, was first pointed out by Maxwell, and the dimensions of the product of k and μ have been known for many years from this relation, namely

$$k\mu = \frac{1}{c^2},$$

where c is the velocity of light. The dimensions of the product are, therefore, $L^{-2}T^2$. It is almost proved by

$L = T$ by Relativity $\therefore$ $k\mu$ is dimensionless

this that both k and μ have some dimensions in terms of L and T. It is most improbable that all of the dimensions of the product fall upon μ alone, and that k is dimensionless. Also, a determination of the dimensions of one of the two quantities automatically determines the other because of this equation. If k has the dimensions of the reciprocal of a velocity, then μ receives the same dimensions, and they each thus represent the same kind of quantity. There has been some speculation in the past as to the dimensions of the ratio of k to μ, for, if this could have been determined, of course each might be found. It is interesting to observe that the values above found make the ratio have no dimensions in terms of L and T, being in this respect like the quantity β, the ratio of two velocities, a pure numeric.

When we examine the expression for the Rydberg constant as given by Bohr, namely

$$K = \frac{2\pi^2 m_0 e^4}{h^3},$$

we find that the dimensions of the left member of the equation are those of the Rydberg constant, T^{-1}, and the right member on the electrostatic system of units has the dimensions

$$T^{-1}k^2$$

and on the electromagnetic system

$$L^{-4}T^3\mu^{-2}.$$

If k is not dimensionless, it ought to be expressed in the above equation. Possibly it is understood that it ought to be there, but this practice of omitting to write it down seems to the author to be pernicious, because it may easily lead to confusion.

There are two equally compelling reasons for believing that the dimensions of mass are those of a velocity,

LT^{-1}, the reciprocal of specific inductive capacity and magnetic permeability. The first is to be found in a new expression for Planck's constant, h, and the second in the gravitational equation, which will be referred to later.

The new expressions for Planck's constant, h, are given in Appendix C, below. The three equivalent values are

$$h = \left(\frac{a_H}{3}\right)^4 \frac{(2K)^3}{c},$$

where a_H is the radius of the hydrogen nucleus, K, Rydberg's constant, and c the velocity of light, and again

$$h = \frac{8^5}{(15k)^4 Kc},$$

how does it check with Lewis expression?

where k is the specific inductive capacity, and finally

$$h = \left(\frac{16}{15k}\right)^4 \frac{e^2}{m_H c^3}.$$

The second and third expressions are derived from the first directly by the use of the Lorentz mass formula and the new expression for the Rydberg constant above given. The second form makes the numerical value of h depend only upon the numerical values of the Rydberg constant, K, and the velocity of light, c, since the specific inductive capacity, k, is numerically unity. Both of these constants are known with exceptional accuracy. The resulting numerical value, taking $K = 3.290 \times 10^{15}$ and $c = 3 \times 10^{10}$, gives

3.2898

check this

2.9986

$$h = 6.5579 \times 10^{-27}.$$

6.5634 using the best values.

This is in almost exact agreement with Millikan's experimental value of h, obtained from his machine-shop-in-vacuo apparatus designed for the purpose of testing the validity of the Einstein equation, which makes

energy equal to Planck's constant times a frequency. The best value of h obtained by Millikan as the result of all these experiments on the emission of electrons from metals by light of different frequencies was

$$h = 6.56 \times 10^{-27}.$$

If we had used a value of the velocity of light slightly less than 3×10^{10}, which would be more exact, we would have obtained a slightly larger theoretical value of h, and a value in almost exact agreement with Millikan's experimental value, since it differs as given by about two units in the fourth significant figure. There seems to be some difference of opinion among authorities as to the exact value of the velocity of light, and, hence, the even number, 3×10^{10}, has been adhered to.

Millikan has published a value for $h = 6.547 \times 10^{-27}$, as representing the most probable value of h when all sources are taken into account, has given some weight apparently to the experimental results of others, and has struck a mean value. So great a difference as that between 6.56 and 6.547 would make an easily perceptible difference in the slope of the straight line from which he derived his own experimental value. It is the author's opinion that there was no gain in the accuracy of the value of h by abandoning the best result of his own experiments and giving so much weight to the results of others.

The dimensions of h are those of energy divided by a frequency, that is, multiplied by a time. Hence these dimensions may be classed with mechanical units rather than electrical and magnetic units, the distinction being that the former do not involve specific inductive capacity and magnetic permeability. The dimensions of energy are L^2MT^{-2}, and multiplying by a time, the dimensions of h must be L^2MT^{-1}.

this makes mass dimensionless, on relativity

The first two expressions for the value of h given above leave no doubt as to the dimensions, for the dimensions of K, c and a_H do not involve specific inductive capacity or magnetic permeability in any way. The dimensions of these first two forms are clearly L^3T^{-2}. If these dimensions are equated to those of h, we have

$$L^2MT^{-1} = L^3T^{-2}.$$

This can only be a true relation if the dimensions of mass are LT^{-1}, that is to say, those of a velocity.

The third expression shows that the dimensions of h are the same as those of the square of the electrical charge of the electron, e. For the m_Hc^3 in the denominator has the dimensions of the fourth power of a velocity, and the k^4 is the reciprocal of this, thus making the whole denominator dimensionless in terms of L and T.

As illustrating the importance of always expressing the specific inductive capacity, some consideration is given to the use of a new system of units of length and of time instead of the present centimeter and the second. The second is not a natural unit of time as applied to atoms, depending as it does upon the rotation of the earth, neither is the centimeter a natural unit of length. The new units considered are the time of one revolution of the electrons in the hydrogen atom, $1/2K$, and the distance traveled by light in this time. On this system of units the velocity of light becomes unity instead of 3×10^{10}, and twice the Rydberg constant becomes unity. Specific inductive capacity has the numerical value 3×10^{10} instead of unity as in the C.G.S. system. The importance of not omitting to express it is, therefore, very evident.

The above expression for the Rydberg constant, namely

$$2Ke^2 = m_Hc^2,$$

makes e^2 not only numerically equal to the mass of the hydrogen atom, since $2K$ and c are each unity, but the third expression for h above, since $e^2 = m_H$ numerically, and $c = 1$, makes h equal to

$$h = \left(\frac{16}{15k}\right)^4 \text{ numerically}$$

but not dimensionally. And, since k on this system is equal to the velocity of light on the C.G.S. system, this expression shows that the numerical value of Planck's constant is made to depend upon an accurate value of the determination of the velocity of light only. The numerical value of h thus obtained, taking $k = 3 \times 10^{10}$, is

$$h = 1.598 \times 10^{-42}.$$

To convert this over into the C.G.S. system of units only requires an accurate knowledge of the Rydberg constant. This is the same result as shown in the second expression for h above, which makes it depend only upon K and c.

Had we always used the units now under discussion for length and for time, we would have been inclined to omit the c and the $2K$, as being unity, from these equations in a manner exactly analogous to the tendency to omit the specific inductive capacity on the C.G.S system because it is unity. This example has served not only to make this apparent, but it has also pointed out some new relations, that the mass of the hydrogen nucleus is numerically equal to the square of the charge of the electron, and again numerically equal to the energy content of the nucleus itself, since we may regard m_Hc^2 or $2Ke^2$ as this energy content, the dimensions of energy being those of the cube of a velocity. Mass is not, however, dimensionally the same as energy, the latter being the cube of the former.

In a later section of this work an expression has been derived from the Lorentz electromagnetic equations for the Newtonian constant of gravitation, which assumes the very simple form

$$k = \frac{1}{3}\frac{h^2}{e^2 m_0},$$

the k here being a different k from specific inductive capacity, namely the Newtonian constant. The Lorentz equations lead to a form of force-equation which may be represented as follows,

$$F = Ce_1e_2r^{-2},$$

in which C is some constant, simply a numeric without dimensions. The dimensions of a force on the left side of this equation are LMT^{-2}, but the dimensions of the right member on the electrostatic system are $LMT^{-2}k$, which is not a force unless the k is dimensionless in terms of L and T, which we are not prepared to admit. Now, it has been found that, if the quantities on the right of this equation are multiplied by the mass of the electron, m_0, we obtain numerically a value which makes the magnitude of the force equal to the force of gravitation, and this has led to the obtaining of the above value for the gravitational constant. As the equation stands, the right member requires to be multiplied by some kind of physical quantity that has dimensions in terms of L and T in order to make the two members agree in dimensions. And, since we have found that the mass of the electron satisfies the numerical requirements, it is reasonable to suppose that it also satisfies the dimensional requirements. After multiplying by a mass, the dimensions of the right member become

$$LMT^{-2}Mk,$$

and the value agrees with a force numerically, which has

the dimensions LMT^{-2}. It is natural to conclude that Mk has zero dimensions in terms of L and T, and that the dimensions of mass are the reciprocal of those of specific inductive capacity. Taking the latter as the reciprocal of a velocity, we may consider mass as a velocity of something and give it dimensions in terms of L and T, namely LT^{-1}. We have thus been led to conclude that neither k nor mass are fundamental units, but that each may be expressed in terms of L and T. Using these values of k and M in terms of L and T, a new table of dimensions has been constructed, which may be called the space-time system of units. This is given in (202½) in the text.

This table itself constitutes an argument in support of the theories that have led to it. Quantities that have already been suspected to be of exactly the same nature receive the same dimensions in the new system of units. For example, quantity of electricity has the same dimensions as quantity of magnetism; electromotive force the same as magnetomotive force; the coefficients of self and mutual induction the same dimensions as electrical capacity; electric force the same as magnetic force. If it may be assumed that we now possess a correct system of the dimensions of units in terms of length and time only, it will prove to be a powerful tool for the proper examination of physical quantities.

A gravitational equation has been obtained (201) by means of an application of electromagnetic theory as applied to the normal state of atoms while not radiating energy, which represents all the laws contained in Newton's statement, but which attributes the cause of the force to the electrons themselves in their motion about the nuclei of their respective atoms. This equation not only gives the approximate magnitude of the gravita-

tional force and leads to the simple expression above given for the Newtonian constant, but it shows that the force is always an attraction obeying the inverse square of the distance law, and that it is proportional to the product of the masses of the two bodies, and it also shows that the attraction is independent of the orientation of the two bodies, whether they be crystals or any other form of matter, — solid, liquid or gaseous.

For those who have followed the author's work through the published articles in the physical journals, it seems to be required to refer to a criticism of this particular phase of the subject that has been made by G. A. Schott. This matter will be found discussed in the text, and it is believed that the criticism has been fully met. It is not necessary to repeat the arguments here, but it may be said that the point of the criticism centered upon the question whether it was legitimate or not to assume that the Doppler factor,

$$A = \frac{\delta t}{\delta \tau} = 1 - \frac{\mathbf{q}_2 \cdot \mathbf{R}}{cR},$$

is sensibly equal to unity when the time average of the force is taken as the electrons circulate about their respective nuclei. On the assumption that it is sensibly equal to unity, Schott has verified the author's conclusions. But, on the assumption that it is equal to the expression just given, he has shown that the result is different. In each case, however, the resulting force obeys the inverse square of the distance law. The result of Schott does not represent the gravitational force in any other respect, while the author's result does represent it in a very complete manner. The recent work of Saha[1] has made it evident that the $\delta \tau$ in the Doppler

[1] Mega Nad Saha, *Phil. Mag.*, Vol. 37, No. 220, April, 1919, p. 347. *Phys. Rev.*, Vol. XIII, N. S., Jan. 1919, p. 34; March, 1919, p. 238.

factor should not refer to time only, but to each of the four coördinates in a generalized Minkowski space, and there are strong grounds for the belief that this Doppler factor will have to undergo a modification in any revised new form of electromagnetic theory. If any change is made in this, the work of Schott in using the value just as it stands is of little value. It seems most probable that a similar multiplying factor will have to come into the second term of the expression for the Doppler factor, if, indeed, it is to be called the Doppler factor any more, of the same nature as the factor that the author has found to be required for the whole force itself, namely that of the mass of the electron. This modification will make the factor sensibly equal to unity, and the results obtained on the assumption that it is unity, which the author made, are of considerable interest because they result in an exact expression for the gravitational laws.

An application of the gravitational equation to gross matter has led to the expression for the mass of a body as follows,

$$m = \frac{e^2}{h} m_0 \Sigma \beta^2.$$

The summation is to be extended to every electron in every atom of the body. The dimensions of this expression are correct, for $\Sigma\beta^2$ is a numeric without dimensions, and the ratio e^2/h has the dimensions $LT^{-1}k$ on the electrostatic system. Putting k as the reciprocal of a velocity, this becomes dimensionless on the space-time system of dimensions. Hence m has the dimensions of m_0 alone, and represents a mass.

It is shown that this expression for the mass of a body, which is derived primarily from its weight, is equivalent to the expression obtained by summing up the total number of nuclei of the atoms in the body, the mass

really residing in the nuclei. It is well known that the weights of bodies are strictly proportional to their masses; but the two physical concepts of mass and weight are not the same and should be carefully distinguished. If we could stop all of the electrons from coursing around their orbits, the weight would vanish but the mass would not, and, since we cannot change the velocities of the electrons, the weights and masses remain proportional. Or rather, if we could stop them, the atoms would disintegrate and cease to exist as atoms and the body would be recognized no longer.

A further application of the gravitational equation to find the weights of rings of electrons on the earth's surface makes the weights of these rings depend chiefly upon the number of electrons in the ring whether the ring happens to be in one kind of an atom or any other kind. This has led to the conception that it may be possible to find the particular combination of rings of electrons that exist in the various kinds of atoms, for the sum of the weights of the rings must equal the weight of the atom, and, knowing the weights of the rings, we may find a combination that gives the proper weight of the atom. A table (223) has been given of the combination of rings thus found. The method is less uncertain when applied to the elements of low atomic weight, but it has been extended clear through the periodic table of the elements, including uranium, largely because the scheme of the combinations seems to be revealed by the elements of low atomic weight. The scheme indicates that atoms are made up of rings of four electrons in much greater numbers than rings of any different number. For example, in the table as given the total number of the rings of four in the seventy elements is 1470, as compared with the next largest number 185 rings of two electrons.

It is not contended that all of these combinations of rings are correct and will never be subject to change, but the main feature of the table, the great preponderance of the numbers of rings of four, may be tested by means of the gravitational equation. For, in any mixed mass of matter, such as the earth for example, it is necessary that the average speed of a single electron shall be very close to the speed of an electron in a ring of just four electrons. By writing down the gravitational equation for the earth as one body and for a single hydrogen atom on its surface for the other body, it happens that the only unknown quantity in the resulting expression is the sum of the squares of the speeds of all the electrons in the earth, which quantity may, therefore, be found. When this is divided by the total number of electrons in the earth, which is also known because it is equal to the mass of the earth in grams times the number of electrons per gram, we thus find the average speed of a single electron in the earth. The number of electrons per gram is known to be approximately equal to the Avogadro constant. The result of this calculation gives the average velocity of an electron in the earth as .0071. The theoretical velocities of electrons in rings are for a ring of two, .00364, a ring of three, .00546, a ring of four, .00728, of five, .0091, and a ring of six, .0109. All of these velocities are in terms of the velocity of light as being unity. The agreement of the velocity of the average electron in the earth with the velocity of an electron in a ring of just four electrons is very close. Rings of three and of two should reduce this average somewhat below that of a ring of four, and we see that .0071 is slightly less than .00728.

It is pointed out that the mass of the earth in grams came into both expressions used in the above calculation, namely in the sum of the squares of the velocities of all

the electrons in the earth and the number of electrons in the earth, so that when we divided the one by the other the mass of the earth in grams canceled out, and no error was, therefore, introduced by any uncertainty in the numerical value of the mass of the earth in grams. This fact points to the conclusion that the result obtained is very general and would be true of any body whatever, the planets and the sun just as well. This is in later sections shown to be true, and considering all of these matters there are strong grounds for thinking that we have established in a fairly positive manner the proposition contained in the atomic weight table (223), that the great majority of the total number of rings of electrons in atoms is just the ring of four electrons.

II

IN presenting new ideas on any subject it is natural to draw a close comparison between the new and the old, for the new conceptions would not be required if existing theories were entirely adequate and stood in complete harmony with the experimental facts as we know them. The places where the prevailing theory seems to be deficient require to be pointed out, so that it shall appear by comparison how completely these difficulties are removed by looking at the matter in a new way. Heretofore attention has been given to atoms when they are neither radiating nor absorbing energy from without, which is often referred to as their steady states, because in this condition it has been held that electromagnetic theory as applied to moving charges of electricity ought to be applicable to the electrons in atoms while in this state. All matter has thus been divided into two great classes, according to whether its atoms are or are not radiating energy.

In the subject immediately before us, however, attention is given for the first time to the atoms in their second state while absorbing and radiating energy, and we shall begin with a brief review of some of the conceptions prevailing to-day as to the structure of the atom. The first theory of atomic structure which had a definite character was offered by Sir J. J. Thomson at a time not long after his discovery of the separate existence of the electron. In this he postulated that each atom consisted of a positive sphere of electrification of

fairly large dimensions within which a number of electrons having a negative charge were circulating in orbits. His reason for assuming the existence of this positive sphere of electrification must have been to provide the means for retaining the negative electrons within the atom, for by this assumption he secured equilibrium for the negative electrons, which are supposed to repel each other.

The electromagnetic theory as applied to an electron itself by several independent investigators, and in particular by H. A. Lorentz, whose so-called "solid electron" has received the most attention, has pointed out what is known as a "mass formula" for the several forms of electrons. The mass formula for the Lorentz electron is as follows:

$$\text{Transverse mass} = \frac{4}{5}\frac{e^2}{c^2a}(1 - \beta^2),^{-\frac{1}{2}} \quad . \; . \; . \quad (1)$$

$$\text{Longitudinal mass} = \frac{4}{5}\frac{e^2}{c^2a}(1 - \beta^2)^{-\frac{3}{2}}. \quad . \; . \; . \quad (2)$$

These expressions tell us that the mass of the electron is a function of its velocity, βc, and that mass is a vector quantity depending upon the direction being considered. That is to say, it differs in different directions. However, when the velocity is small compared with that of light, the mass approaches the constant value

$$m_0 = \frac{4}{5}\frac{e^2}{c^2a}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

where e denotes the electrical charge, c the velocity of light, and a the radius of the sphere of the electron. For values of the velocity of the electron greater than say one tenth of the velocity of light, this theory shows an appreciable increase in the mass with increasing velocity, the limit being an infinite mass when $\beta = 1$.

The experiments of Kaufmann in measuring the mass of the electron for various values of its velocity up to very high velocities showed a variation in the mass with the velocity in fairly good agreement with the results obtained from electromagnetic theory above mentioned. This good agreement between electrical theory and observation has led to the belief that all ordinary mass has an electromagnetic character, and that there is but one kind of mass. Granting that this may be a fact, it becomes obvious by means of equation (3) that we may find the radius of the Lorentz negative electron, *a*. It is

$$a = \frac{4}{5}\frac{e^2}{c^2 m_0}, \qquad (4)$$

or numerically

$$a = 2.25 \times 10^{-13} cm. \qquad (5)$$

The chief reason for alluding to this well-known history is that this mass formula (4) shows that the radius of the electron is inversely proportional to the mass for slow velocities. If, therefore, the mass of the positive nucleus of the atom is many times greater than the mass of the negative electron, as it is known to be, then its radius should be proportionally smaller than that of the negative electron instead of larger than it.

These ideas are contrary to the conception of an atom as first suggested by Sir J. J. Thomson above mentioned, who made the positive charge of the atom occupy a larger volume than the electron many times over. This reason, and the fact that it was proved by experiments on the scattering of alpha particles that the central positive charge of atoms must occupy a very small space indeed, led Sir Ernest Rutherford to propose a new atomic theory, in which the positive charge of the atom is supposed to be of extremely small dimensions, the electrons not being

within the positive charge but outside of it, circulating about it in orbits that are large in comparison with either the nucleus or the electron itself.

This Rutherford theory was, however, beset with theoretical difficulties from the beginning because it could not be shown by the accepted form of electromagnetic theory that there existed any possible orbits for a group of electrons which would be stable orbits. And thus it has come about that men have come to believe in a form of atom that the accepted form of electromagnetic theory cannot sustain, and yet this very form has been forced upon us by certain other applications of electromagnetic theory as applied to the electrons themselves, which has been outlined above. It must be that electromagnetic theory has certain good features; that it represents a truth at some points at least, but that it is deficient and unsatisfactory in other points, so that it becomes a puzzle when it is safe to use it and when it is unsafe. That some modification of electromagnetic theory is possible, which will bring it into line with these atomic phenomena, makes a very strong appeal to right reason.

It was at the time that this Rutherford theory of the atom was laboring under these difficulties that Dr. N. Bohr came to the rescue by providing the means whereby these unstable electrons might find stability. The rescue was not, unfortunately, effected by any modification of electromagnetic theory, which might have again given us a sense of security as being based upon a solid foundation that was understandable, but was secured by bringing to bear upon the Rutherford atomic theory the ideas of Planck and of Einstein, which admittedly are not founded upon electromagnetic theory, but are chiefly based upon experimental evidence. This evidence is so strong that it

is difficult to disbelieve in the truth of the assertions contained in what is now known as Planck's quantum theory, and indeed there is no desire to disbelieve in this theory; for the only desires of the true investigator are to learn the truth whether he yet understands the reason for it or not. At the same time there exists a very strong and natural desire that these truths shall some day be shown to be the result of a more comprehensive and general electromagnetic theory than we now possess. We may say, therefore, that it now becomes one of the chief and legitimate aims of the investigator to seek to discover some modified form of theory which may harmonize all of these various phenomena that now have no interpretation in terms of the accepted form of it.

This very conclusion compels us to doubt the general applicability of the present form of the theory in atomic phenomena. At the same time the present theory gives a good account of itself at certain points, as we have pointed out above in one instance, namely by showing that the positive nucleus of the atom should occupy a very small volume. We are not justified in throwing it all aside, for it has proved itself to be correct in too many instances, and, moreover, without it we are entirely at sea, and everything seems confused and without any theoretical basis. We must use the theory in part and learn to distinguish when possible between those cases where it is applicable and where it is not.

Hereafter it cannot, therefore, be regarded as illegitimate or even strange when certain assumptions are made that are not in strict accord with the current form of electromagnetic theory, for it is only by such attempts that there is hope eventually of discovering some modification of the present theory that will harmonize everything.

We shall content ourselves by giving very briefly a statement of some of the chief features introduced by Dr. Bohr into the theory of the Rutherford atom, by which the solution of the question of atomic structure assumed a definite form. Starting with the simplest atom, that of hydrogen, he has concluded that in its normal neutral condition there is a single nucleus of charge plus *e* and a single electron of charge minus *e* circulating around the nucleus in an orbit. When the atom is neither absorbing nor radiating any energy this orbit is supposed to have a circular form, but strangely enough, the radius of it may have a theoretically infinite number of values at different times, and the electron may be stable in any one of this large series of different-sized orbits. The actual velocity of the electron is supposed to differ in each of these so-called stable orbits, and, therefore, the kinetic energy of the electron differs in every one of them. In each instance, however, the attraction between the electron and the nucleus is supposed to obey the inverse square of the distance law, and the velocity in each orbit is so adjusted that the centrifugal force of the electron due to its mass is exactly balanced by the force of attraction toward the nucleus.

Next, as to the manner in which such an atom emits its energy in the form of vibrations that correspond exactly with the vibrations that hydrogen atoms are known to emit by observations of the spectrum of hydrogen. If such an atom should receive energy from some external source, it is supposed that the electron suddenly goes outward from the nucleus, moving from the orbit in which it then happens to be to some one of the other larger orbits, which one depending upon how much energy has been received. After all the energy has been received that is coming to it on this one occasion it is

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then free to radiate energy again by dropping back to some smaller orbit. There is a difficulty here in seeing why it should leave the larger orbit at all if it is in equilibrium there, after it has been brought to it by the receipt of energy. And again, it is difficult to see in which one of the various possible stable orbits between its outermost position and the smallest possible orbit it will stop in its course back toward the nucleus. It is true that it will radiate the most energy if it goes clear through to the last or smallest orbit, which is considered the most stable position; but, if it does this every time it is displaced, then the size of the orbits in a normal mass of hydrogen gas, while not radiating energy, must all be alike and equal to the smallest possible orbit. If this is the case, there seems to be no possible utility in postulating a large number of stable orbits, if the electron is never to stay in any one of them. For the return to the nucleus must begin immediately after it has reached its maximum distance, and there would be no time in which it remains stable in its larger orbit. If, on the other hand, it does remain in its larger orbit after receiving the initial energy that drove it there, and waits there until the next disturbance is received, it is true that this disturbance might subtract energy instead of add it, and thus assist in bringing it back to any smaller orbit. Admitting this to be the case, it also follows that the new impulse might drive it further away, and hence it is probable, on the theory of chance, that one first impulse may drive it from the first to the second orbit, and that it will remain there for a time; then a second impulse might follow that drives it to the third orbit, and this again be followed by one which drives it one further on, and so forth. The series of successive impulses thus at the last drives it completely away from the nucleus ionizing the gas.

The ionizing voltage required to accomplish this on the part of the bombarding electrons will, therefore, be no greater than the greatest of the links in this chain of happenings. The numerical value of the ionizing voltage given by this hypothesis, however, does not agree with the results of experiments on hydrogen. There is scant experimental support for the assumption of a large series of stable orbits.

Now as to the frequencies of vibration at which this radiated energy is dissipated. When the electron changes over from one of these larger orbits to a smaller one it also changes from one frequency of revolution to another around the nucleus. These frequencies of revolution are such according to the theory that the difference between the frequencies of revolution of the electron in any two of the orbits whatever is strictly proportional to some one of the frequencies which are known to be emitted by hydrogen as found in the observations of the spectrum of hydrogen. The theory assumes, therefore, that but one single harmonic frequency is emitted each time an electron changes over from a larger to a smaller orbit. That is to say it is assumed that there exists a uniform harmonic vibration of something during the time that the electron is changing orbits. Nothing is said as to the path by which the electron makes the change. It is impossible that the path should be a simple circular path, for no circular arc can be drawn connecting the outer orbit with the inner one without some sudden transition in the direction of motion, and we have the hypothesis inherent in the theory that the electron is emitting a perfectly uniform and fixed frequency while it moves in some form of path that cannot be simple.

When this, the greatest and most important objection to the Bohr theory, is pointed out to an advocate of this

theory, I have heard the reply that this is "the assumption." It is admitted that we do eventually come to some last thing in the course of explaining things in terms of other things, which itself cannot be explained in terms of anything. There should, however, be some choice allowed in selecting our last trench for a final stand. It must stand certain tests and be at least reasonable. If something less objectionable can be substituted for the above almost unthinkable assumption of a uniform vibration emitted during a complex motion of an electron, it may perhaps get a hearing.

In the theory about to be described we are left in no uncertainty as to the source of the vibrations which cause the lines in the spectrum of hydrogen. They are due to the very motion of the electrons themselves in returning to the nucleus again after displacement by the receipt of energy from the external source. The forms of the paths followed by the electrons may emit a whole series of frequencies at once, all of which are, however, to be found in the observed spectrum. But this statement anticipates the logical presentation of the theory, which is based chiefly upon an application of the electromagnetic theory to the problem.

III

SOME light is thrown upon the subject by a consideration of the mechanical force that one single moving electrical charge exerts upon another according to the prevailing form of the electromagnetic theory. Let us at first suppose that the motion of one of the two charges being considered is circular motion, and that the other charge is at rest. We may form a definite picture by imagining that the stationary charge is positive and represents the nucleus of some atom fixed in a photographic plate ready to receive the radiation from a small amount of hydrogen gas confined in a vacuum tube some distance away. The charge moving in the circle may then represent a single negative electron in one of the atoms of the hydrogen gas. Whether or not the photographic plate shows anything upon development will depend upon the mechanical force that has acted upon this one atomic nucleus, which really represents them all and thus represents the whole photographic plate.

The equation expressing this instantaneous force acting upon one revolving electron due to a second electron has been developed in full, and the so-called electric component of it published in equations (48), (49), and (50), pages 453, 454, of the *Physical Review* for June, 1917. Fortunately this equation has been checked by Dr. G. A. Schott in an article in the *Physical Review* for July, 1918, where, on page 23, he remarks, "The following investigation is based on Crehore's equations for the electric part of the mechanical force (*loc. cit.*, pp. 453, 454), which

have been verified, except some obvious misprints, e.g., a_2 for a_1 in the last term of (49)." By making the radius of the orbit of the first electron, a_1, equal to zero, so that the electron is brought to rest, and by changing the sign of the equation so as to make the stationary charge positive instead of negative, this equation is strictly applicable to the case we have chosen as an example above. The distance between the positive nucleus at O in the photographic plate and the center of the orbit of the revolving electron O' in the hydrogen gas is supposed to be fixed or constant and is represented by r. An inspection of the equation referred to shows that there are some terms in it which vary as the inverse first power of r, and others as the inverse square and higher powers.

Now, any term which varies as the inverse first power of the distance becomes immensely greater than terms which vary as the inverse square or higher powers of the distance if the distance r is taken large enough. Let us pick out, therefore, from this equation only those terms which vary as the inverse distance and write them down separately. It must be stated, however, that we are at liberty to choose the i, j, and k axes in any directions we please because the charge e_1 is now supposed to be stationary, and it can, therefore, make no difference in the force upon it how this electron is oriented. Indeed it lost its power of orientation as soon as it ceased to revolve in an orbit. Let us, then, take the k-axis along the line joining the centers, namely the line OO'. The angle α is then the angle between the direction of this k-axis and the axis of revolution of the negative electron. The coördinates, x, y, and z, of the equation locate the position of the point O' with reference to O, along the i, j, and k axes respectively. Since we have

now located the point O' on the k-axis, both x and y are zero. Hence, putting $x = y = a_1 = 0$ in this equation, and retaining only terms varying as the inverse first power of r, we obtain as the electric force

$$e_1 E_i = \frac{e_1 e_2}{RA^3} \frac{\beta_2^2}{a_2} (\cos \alpha) S_2 i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$e_1 E_j = \frac{e_1 e_2}{RA^3} \frac{\beta_2^2}{a_2} C_2 j, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$e_1 E_k = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The letters S_2 and C_2 are abbreviations for the following:

$$S_2 = \sin \left[\omega_2 \left(t - \frac{R}{c} \right) + \theta_2 \right], \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$C_2 = \cos \left[\omega_2 \left(t - \frac{R}{c} \right) + \theta_2 \right], \quad . \quad . \quad . \quad . \quad (10)$$

these quantities being functions of the time. The letter A stands for the quantity

$$A = 1 - \frac{q_2 \cdot R}{cR}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and this may be considered to be equal to unity for our present purpose, since q_2 is small compared with c; that is, the velocity of the electron e_2 is small compared with the velocity of light.

In these equations it is found that there are no terms in the k-component of the force that vary as the inverse distance, and hence the force in (8) is put equal to zero, the meaning being that the force is very small by comparison with the forces along the i and j axes, and that this small force varies as the inverse square of the distance instead of the inverse first power. When the distance OO' is large compared with the radius of the orbit of the revolving electron at O', then the instanta-

neous distance R between the point O and the instantaneous position of the electron becomes very approximately equal to r, the distance between centers which is fixed, and we may write very approximately the sum of the three forces (6), (7) and (8) as follows:

$$e_1E = \frac{e_1e_2}{r}\frac{\beta_2^2}{a_2}[(\cos\alpha)S_2i + C_2j]. \quad . \quad . \quad . \quad (12)$$

Let us examine this equation and put into words some of the statements that it implies. The quantities outside of the bracket are constants. These are the fixed distance between O and O', namely r; the fixed electrical charge on the electron e_2 which is revolving, and the fixed electrical charge on the stationary nucleus e_1; the constant speed of the moving electron β_2, expressed in terms of the velocity of light as a unit, and the fixed radius of the orbit of e_2, namely a_2. Within the bracket the S_2 and C_2 are simple harmonic functions of the time, according to (9) and (10) above, regarding R as equivalent to r. There remains only the angle α, which denotes the angle between the line OO' and the axis of revolution of the electron e_2. When this angle is zero, the axis of revolution coincides with the line OO', and the plane of the orbit is then perpendicular to the line joining centers, and the orbit, when viewed from O, appears as a circle. In this position $\cos\alpha = 1$, and the force expressed by (12) is then a purely circular force made up of two harmonic components at right angles to each other. This whole force lies in the i-j plane perpendicular to the line OO', and is also in this instance parallel to the plane of the orbit of e_2.

If the plane of the orbit of e_2 were turned through a right angle so as to contain the line OO', the orbit when viewed from O would appear as a straight line instead

of a circle. The angle α would be a right angle, and its cosine be equal to zero, so that the i-term vanishes from equation (12). The force then becomes a simple harmonic force, still being in the plane perpendicular to OO', however, but in one straight line only, parallel to the axis of j. For any position of the orbit of e_2 intermediate between these two extreme positions just supposed, the cosine of α is less than unity and greater than zero, and the orbit, as viewed from the point O, appears as an ellipse. The force upon e_1 then has two harmonic components at right angles to each other of the same period but different amplitudes, the amplitude along the i-axis being less than that along the j-axis. The force then becomes an elliptical force still acting in the plane perpendicular to the line OO'.

Now the equation expressing circular motion of the electron e_2, from which this force equation has been derived, is as follows:

$$r_2 = a_2[(\cos\alpha)S_2 i + C_2 j + (\sin\alpha)S_2 k]. \quad . \; . \; . \; . \; . \quad (13)$$

By a first differentiation of this with respect to the time, the vector velocity of the electron is obtained as follows:

$$q_2 = a_2\omega_2[(\cos\alpha)C_2 i - S_2 j + (\sin\alpha)C_2 k]. \quad . \; . \; . \; . \quad (14)$$

And by a second differentiation, the vector acceleration of the electron e_2 is as follows:

$$f_2 = -a_2\omega_2^2[(\cos\alpha)S_2 i + C_2 j + (\sin\alpha)S_2 k]. \quad . \; . \; . \quad (15)$$

The coefficient of (15) may be written in terms of β_2 for comparison with (12) above because of the relation of definition

$$\beta_2 = \frac{a_2\omega_2}{c}, \qquad \text{or} \qquad a_2\omega_2^2 = c^2\frac{\beta_2^2}{a_2}. \quad . \; . \; . \; . \; . \; . \; . \quad (16)$$

Whence, we have as an equivalent of (15)

$$f_2 = -c^2\frac{\beta_2{}^2}{a_2}[(\cos\alpha)S_2i + C_2j + (\sin\alpha)S_2k]. \quad . \quad . \quad . \quad (17)$$

It is at once apparent from a comparison between this acceleration and the expression for the mechanical force upon the stationary nucleus, (12), that the portion within the bracket is exactly the same in the two expressions, except that the k-component is missing in the force equation. Except for a certain constant multiplier the expression for the force is evidently equal to the portion of the acceleration which is resolved in the i-j plane, perpendicular to the line OO'.

Let us, therefore, denote by f_{ij} the sum of the i and the j components of the acceleration only, and write

$$f_{ij} = -c^2\frac{\beta_2{}^2}{a_2}[(\cos\alpha)S_2i + C_2j]. \quad . \quad . \quad . \quad (18)$$

If this is multiplied by the quantity e_1e_2/c^2r, we obtain

$$\frac{e_1e_2}{c^2r}f_{ij} = -\frac{e_1e_2}{r}\frac{\beta_2{}^2}{a_2}[(\cos\alpha)S_2i + C_2j]. \quad . \quad . \quad . \quad (19)$$

But this is exactly equal and of opposite sign to the expression for the mechanical force in (12). Hence we may equate the first members and obtain the equation

$$e_1E = -\frac{e_1e_2}{c^2r}f_{ij}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

In words this equation states that the mechanical force e_1E, acting upon the stationary nucleus e_1 of the atom in the photographic plate at O, due to the revolving electron about the point O' in the hydrogen gas, is proportional to the acceleration of the revolving electron when resolved in the plane perpendicular to the line OO', but that it has the opposite sign or direction. It does

not matter what the acceleration of the moving electron may be in the direction of the line of centers OO' so far as the mechanical force is concerned, for it is of no effect.

Thus far we have dealt with but one component of the mechanical force, namely the electric component, $e_1\boldsymbol{E}$. To this must be added the magnetic component in order that the result shall be perfectly general. The complete expression for the mechanical force in the current form of electromagnetic theory due to Larmor and Lorentz is as follows:

$$\boldsymbol{F} = e_1\left(\boldsymbol{E} + \frac{1}{c}\, \boldsymbol{q}_1 \times \boldsymbol{H}\right), \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where $\boldsymbol{q}_1$ is the velocity of the first charge upon which we are getting the force. In the present example this charge is the stationary nucleus of the atom in the photographic plate, which is supposed to have no velocity, so that $\boldsymbol{q}_1 = 0$, and the magnetic component of the force is, therefore, zero. The result given above in (20) is, therefore, perfectly general, and requires no modification, the whole mechanical force being due to the electric component of the force.

Let us next discuss this equation (20) more fully. It follows directly from this result that the statement is true whether the motion of the electron in the hydrogen gas is circular or not, namely, that the force upon the fixed nucleus is proportional to the acceleration of the moving electron resolved in the plane perpendicular to the line OO'; for, had we assumed at the beginning that the motion of the moving electron was compounded of two simple circular motions having independent periods and amplitudes, we should have arrived at the conclusion that the force upon the stationary nucleus is merely proportional to the sum of the two independent accelera-

tions of the electron resolved in the *i-j* plane. From this we may immediately infer that the theorem is general for any kind of motion whatever of the moving electron, for this complex motion may, by means of the well-known theorem of Fourier, be resolved into a series of simple harmonic motions and a series of simple harmonic accelerations.

If this is true, it ought to be possible to give a more general proof of it by going back to the fundamental equations of electromagnetic theory without making any assumption as to circular motion. And, indeed, this proof of the theorem has been obtained in this manner directly from the more fundamental equations, but it is not given here partly because of the space required, as well as the necessity for the introduction of the vector notation in which these fundamental equations are expressed. The proof above given is, however, just as general and is easily grasped. It rests, however, upon the assumption that the equation for the force with circular motion of the electrons has been obtained without error from the more general equations. Since this equation has been checked by G. A. Schott, as stated, it is safe to assume that it is correct on the premises.

IV

WE are now in a position to examine into some of the consequences of this theorem, which may be regarded as established, as applied to the case before us, namely the hydrogen gas radiating its energy so that a portion of it is received by the atoms in the photographic plate. We shall postulate at the beginning that what we see when the plate is developed after exposure is in a sense merely a record of the energy that atoms in the plate received during exposure to the radiation, and need not concern ourselves at present with the obscure processes by which this energy is revealed to us through the process of development of the plate. We shall also consider that this energy is some function of the mechanical force acting upon the nuclei of the atoms of the plate during exposure. Each atom in the plate is, of course, acted upon by a large number of the electrons in the distant hydrogen gas, and the force that any one atom in the plate experiences may be regarded as proportional to the resolved sum of the accelerations of all the electrons in the hydrogen gas which are brought to bear upon it.

Let us, first, therefore, give some consideration to the normal state of this gas, in which it is supposed that all the orbits of the electrons are true circular paths. If these orbits all have the same radius and if the revolution of the electrons is at the same speed in all, then the sum of the forces due to them all acting upon the atom at *O*, being a vector sum and approximately in one plane per-

pendicular to the line joining the photographic plate and the hydrogen gas, would probably be very small, because the planes of the orbits of the electrons in the atoms of hydrogen are turned in every possible orientation, and there would be a tendency to cancellation of the force. To obtain a rigid proof that the force would be exactly zero under these circumstances it would be necessary to allow for the slight differences in the distances of the centers of the orbits in the hydrogen from the nucleus of the atom at O. The problem under these assumed conditions becomes a statistical question that would require treatment by the theory of probabilities, and presents some difficulty.

The difficulty would be much greater if we should imagine that an infinite number of possible stable orbits exists, each electron having a frequency of revolution corresponding to the particular orbit, in the normal neutral state of the hydrogen. The chance that all of the force would cancel would be far smaller. They should, however, exactly cancel in order to agree with observation; for the normal gas emits no characteristic radiation and does not affect a photographic plate. There is a difficulty here in supposing that a large number of different-sized orbits can exist in the normal state of the gas. And, if they do not exist in the normal state, there is no utility in supposing that they ever exist at all.

All of these difficulties disappear when it is assumed that each hydrogen atom has two electrons instead of one in its normal state, these electrons being located at the opposite ends of a common diameter of the orbit; for, then, the sum of the accelerations of the two electrons in each atom is exactly zero, and of course the total force upon the nucleus of the atom at O is always zero as long as the electrons in the hydrogen follow a purely

circular path. There is then no necessity to attempt a proof of the statistical theorem just proposed for the single electron atom. This fact in itself supplies an additional reason to those obtained from other considerations for supposing that the hydrogen atom has two instead of a single electron. We shall throughout this work assume that the hydrogen atom has two electrons in its normal state.

It has now been shown that hydrogen gas in its normal undisturbed state will produce no force upon any atom in the photographic plate, and will behave as though no energy is being radiated, a circumstance that is in complete agreement with observations. If we now begin to disturb the normal condition of the gas by bombardment with electrons from an external source, or by alpha particles, then the effect may be pictured by supposing that some of the electrons in some of the atoms are driven away farther from the nucleus of their atoms than is normal, following paths that are more complicated than the purely circular orbit, passing out to a maximum distance, which depends upon the amount of energy absorbed, and returning again to the same normal circular motion after a very brief time. Some of the electrons may even be driven completely away from the nucleus, thus ionizing the gas. It is not supposed that every atom in the gas is affected simultaneously and continuously, but that first one and then another is hit, so to speak, its electrons being driven out and returning immediately before the next hit is registered on this particular atom. The apparently continuous bombardment is really distributed among a large number of atoms of the gas, no one of them experiencing a continuous impulsive force.

It is our purpose to inquire into the nature of the paths

described by the electrons in the hydrogen atoms in returning to their normal state after being disturbed. The chief guide must be the known spectrum of hydrogen and an application of the theorem obtained above from electromagnetic theory to the case. Any admissible form of path must be such that only those accelerations are admissible which have periods equal to those in the observed spectrum. In this mode of looking at the subject we are reversing the common procedure, which assumes as a starting point the law of force acting and attempts to work out from it the paths of the electrons. We do not assume any law of force, but leave that to come out as a final product instead of an initial assumption.

Let us first place before us the whole spectrum of hydrogen for constant reference. This element is selected for one reason because the atom of hydrogen is probably the simplest in structure of any of the atoms, and for another because the complete spectrum of hydrogen is known with great probability all the way from the lowest possible frequency up to the highest, as we shall see, although the total number of lines in the spectrum is infinite. Fortunately in the case of hydrogen a mathematical formula has been found that makes it possible to express in extremely simple language all of the infinity of lines in this spectrum. This cannot yet be said to be true of many of the other elements. The spectrum of hydrogen is shown in Fig. 1. The lines shown in this chart have been observed experimentally. The visible portion of the spectrum lies between the points indicated approximately, and only a very limited number of lines in the Balmer series lie in this region. If we knew only the spectrum in the visible region, it is evident that no important generalization could have been obtained from

it. All of the lines that have been observed may be very accurately expressed by the formula

$$\nu' = K\left(\frac{1}{\tau_2^2} - \frac{1}{\tau_1^2}\right). \quad \ldots \ldots \quad (22)$$

In this formula ν' denotes the frequency of the vibration corresponding to the line which is indicated by its wave-length in the figure. K is a constant equal to 3.290×10^{15} and known as Rydberg's constant, which is

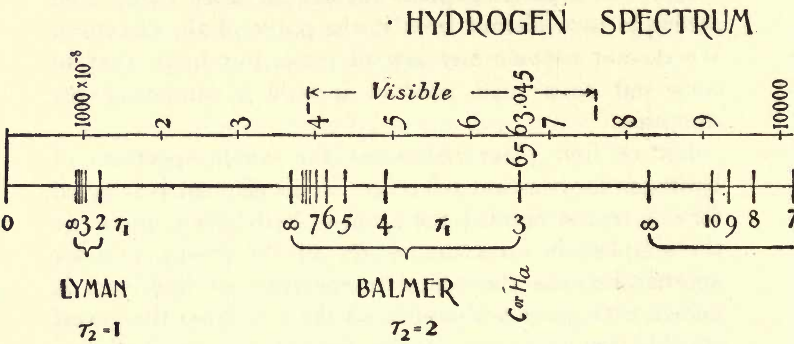

Fig. 1.

one of the most important constants we know of in nature. It appears not only in the spectrum of hydrogen but in the spectrum of all of the other elements whose spectra have been expressed by a formula. τ_2 and τ_1 are simply integers, and may have any values whatever, provided only they are not such as to make the frequency have a negative value.

Let us first look at this formula in the usual way and afterwards consider the modification of it that we shall require in the interpretation of the emission of this spectrum by the atoms of the gas. If we set $\tau_2 = 1$, and then give to τ_1 values 2, 3, 4, etc., to infinity, we obtain the

frequencies of all the lines of the so-called Lyman series, which has the greatest frequencies and shortest wave-lengths and is seen at the extreme left in Fig. 1 cramped up between comparatively narrow limits. If we set $\tau_2 = 2$, and give to τ_1 the successive values 3, 4, 5, etc., to infinity, we obtain the series known as the Balmer series, part of which appears in the visible spectrum in the central portion of the figure, the rest extending into the ultra-violet. If we set $\tau_2 = 3$, and give to τ_1 the

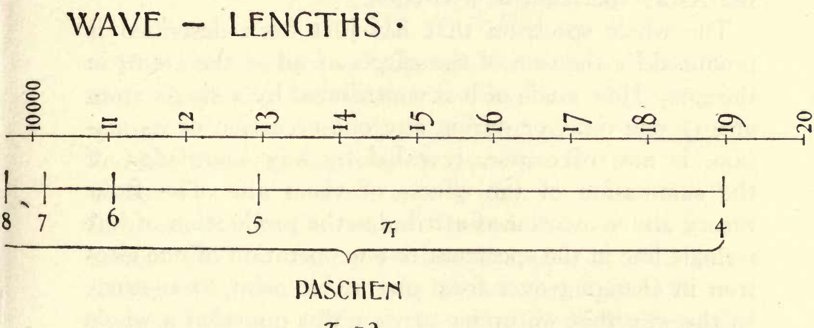

Fig. 1.

values 4, 5, 6, etc., to infinity, we obtain the series known as the Paschen series at the right in the figure. All of this series is in the infra-red portion of the spectrum.

A sufficient number of lines has thus been observed to make it quite certain that the formula (22) is general, that is to say, we may predict from it that, if we should set $\tau_2 = 4$, we should obtain another similar series of lines which have not yet been observed experimentally further off in the infra-red beyond the Paschen series, and so on for values of $\tau_2 = 5$, 6, 7, etc., to infinity. The frequencies grow smaller and smaller and the wave-

lengths greater and greater as we advance in this direction.

The X-ray spectrum of hydrogen has never been obtained directly by the use of these rays. There are grave difficulties in getting the X-ray spectra of the elements of low atomic number. There are reasons to believe that the X-ray spectrum of hydrogen will never extend the spectrum to shorter wave-lengths than those given in the Lyman series, and that in this series we really have the X-ray spectrum of hydrogen.

The whole spectrum that has just been described is produced by the sum of the effects of all of the atoms in the gas. How much of it is contributed by a single atom at any one time, or during any one excursion of its motion, is not, of course, revealed by any knowledge of the summation of the effects of them all. The Bohr theory above mentioned attributes the production of but a single line in the spectrum to one operation of one electron in changing over from one stable orbit to another. In the way that we prefer to view this question a whole series of spectral lines is emitted by the two electrons in the atom in one operation while they are returning again to their original orbit after being displaced. But this series of lines is neither the Lyman, Balmer, Paschen, nor any of the series commonly spoken of. Assuming, for the sake of forming a definite picture, that the two electrons return to the original orbit in a species of spiral paths approaching rapidly at first and then slower and slower as they get closer and closer to the final orbit, it appears that the sum of the accelerations of these two electrons should contain frequencies which change rapidly at first and then slower and slower, approaching zero as they finally attain their steady orbit. The frequencies in, say, the Balmer series, however, are crowded close

together at a head and separated more and more as we go down in the series to lower frequencies. According to this way of viewing the matter, these series are not so well adapted to the case as if we looked at equation (22) in a different way.

The following equation is exactly equivalent to (22) and has the advantage that the spectrum is divided up into series in a manner more suited for our purposes.

$$\nu' = K\left(\frac{1}{\tau^2} - \frac{1}{(\tau + \tau_2)^2}\right) \quad . \; . \; . \; . \; . \; . \quad (23)$$

If in this we set $\tau_2 = 1$, and give to τ the values 1, 2, 3, etc., to infinity we obtain the first lines of lowest frequency of every one of the former series, namely the Lyman, Balmer, Paschen, etc., series. If we set $\tau_2 = 2$, and give to τ the values 1, 2, 3, etc., to infinity, as before, we obtain the second lines of each of the former series. And so on, by setting $\tau_2 = 3, 4, 5$, etc., to infinity, we obtain eventually all of the lines that were obtained before. When $\tau_2 = \infty$, all of the so-called "heads" of the Lyman, Balmer, Paschen, etc., series are obtained.

An inspection of the formula shows that, as τ becomes larger and larger, the two terms within the parenthesis approach each other in value continually, and their difference, therefore, approaches zero in every case, but very slowly. That is to say, the lines in these series are crowded nearer and nearer together at the low-frequency end of the series. The wave-lengths, however, do not appear so crowded since they are proportional to the reciprocals of the frequencies, and an extremely small difference in a small frequency may make a large absolute difference in the wave-length.

As stated above, we are about to discuss possible forms

of the paths followed by the electrons in returning to their original orbit after being displaced. It will, of course, be understood that there is not sufficient data supplied by a knowledge of the spectrum alone to make this problem susceptible of but one solution. We have supposed that the spectrum supplies us with a knowledge of the frequencies only that enter into the sum of the accelerations of the electrons in their motions in the gas while radiating energy. Further experimental data as to the amplitudes of these various harmonic components of the accelerations is necessary. A knowledge of a sum is not as satisfactory as a knowledge of each of its component parts, the accelerations of the individual electrons, because a sum may be made up in an infinite variety of ways. But it is the best knowledge we have at present, and the only way that is open is to make the most reasonable assumption that we can concerning the individual parts that is always in complete accord with the knowledge we possess of the sum, and see whether the results derivable from the assumptions are in complete harmony with the known facts of observation. It is most desirable to have experimental checks at as many points as possible in order that the assumptions made may be tested in a variety of ways. One experimental check is afforded by the agreement between the observed values of the voltages required to ionize the hydrogen gas, that is, to separate some of the electrons completely from their atomic nuclei. It is shown in a later section that the theory here presented gives all of the ionizing voltages to within the experimental error that have been observed in hydrogen gas. The Bohr theory gives no indication of an ionizing voltage at 15.8 volts recently observed. The lowest voltage that the Bohr theory gives is 10.15 volts, but ionization does not

begin until the voltage is above 11, according to the most reliable experimental results. This theory gives the minimum value as 11.132 volts and the maximum value as 15.5 volts, which is in better agreement with the observations.

V

NO apology is, therefore, offered for writing on this subject in a suggestive and general manner. The fundamentally new point in the theory is that a whole series of spectrum lines are produced in one operation of the electrons in returning by a species of spiral paths to their original stable orbit, and that there is but one orbit of a uniform size in the normal hydrogen atom. It is not intended to imply by this assertion that all atoms of one element are always exactly alike in their neutral condition. Where there are a multiplicity of rings of electrons, as in most of the more complex atoms, it is very likely that there are a number of different possible stable configurations dependent upon the order of the rings as we proceed outward from the nucleus. If, for example, a ring of three electrons should interchange places with a ring of four, it is held that this would not affect many of the properties of the atom, such as its weight and atomic number, but that it may affect other properties such as its chemical valency. So long as the weight and atomic number remain unchanged, such an atom is called by the same name, but many atoms having the same name are known to behave differently in combining with other atoms on different occasions. In the case of hydrogen, however, no interchange of rings is possible because there is but one.

It is of course possible that the precise paths followed by the electrons in hydrogen, which we are about to describe, may not be the actual paths, since the problem admittedly has a number of possible solutions on the

limited amount of experimental data now available, but it seems much more desirable to make definite assumptions, which lead to definite results, even if future experimental data shall compel a revision of these assumptions, than it is not to make such assumptions. For, with definite assumptions, it is possible to draw definite conclusions and then to compare these with the facts of observation so far as possible. Unless some contradiction between the assumptions and experimental data is found we will then have at least one possible solution, which is worth something.

Let us, therefore, denote by ν, 2π times the frequency of a particular line in the spectrum. ν is then an angular velocity as follows:

$$\nu = 2\pi K\left(\frac{1}{\tau^2} - \frac{1}{(\tau + \tau_2)^2}\right) \quad . \; . \; . \; . \; . \; (24)$$

It is assumed that in any one motion of the electrons in the hydrogen atom τ_2 is a fixed integer dependent upon the amount of energy that the system has absorbed from an external source, but that τ takes all possible values at once from 1 to infinity. Let us denote by ρ_2 and ρ_1 the position vectors of the two electrons e_2 and e_1 as they move about, and take the hydrogen nucleus as the origin of these vectors. Then the vector velocities of these electrons are denoted by $\frac{d\rho_2}{dt}$ and $\frac{d\rho_1}{dt}$, and the vector accelerations by $\frac{d^2\rho_2}{dt^2}$ and $\frac{d^2\rho_1}{dt^2}$. Let us now assume that the sum of the two vector accelerations of the two electrons in one atom during one excursion is represented by the equation

$$\frac{d^2\rho_2}{dt^2} + \frac{d^2\rho_1}{dt^2} = k_{\tau_2} \sum_{\tau=1}^{\tau=\infty} \left\{ \nu^3 e^{-\nu t} \left[(\sin \nu t)i + (\cos \nu t)j\right] \right\}, \quad . \; (25)$$

in which k_{τ_2} remains constant during the whole motion,

t represents time and ν is the expression in (24). The summation means that we are to write a term like that in the brace for every value of ν corresponding to values of τ from 1 to infinity, so that the equation is an infinite series of terms. Both k_{τ_2} and ν are functions of τ_2, which remains fixed during one excursion, but differs for different excursions on different occasions. Hence the equation is different for each different amount of energy absorbed by the system. According to the theorem above established, the force that these two electrons in the one hydrogen atom exerts upon the nucleus of the atom at O in the photographic plate is proportional to the sum of the accelerations of the electrons resolved in the i-j plane, that is, a plane perpendicular to the line joining the nuclei of the two atoms. With a different constant multiplier this equation (25), therefore, represents the force that the one hydrogen atom contributes during one single excursion to the formation of the spectrum of hydrogen. This is entirely consistent with the observed spectrum because the only frequencies contained in the force equation are those which are observed in the resulting spectrum of hydrogen. These frequencies are, however, infinite in number and consist of all the first lines, say, of the Lyman, Balmer, Paschen, fourth, etc., series, or of all of the second, third, etc., lines of these series on different occasions. A multitude of hydrogen atoms, each receiving a different amount of energy, will accordingly produce the whole spectrum of hydrogen, and the same atom at a later time may give a different series of lines.

It will be noticed that there is no k-component in the acceleration (25). This is merely because the spectrum tells us nothing about it, and the equation merely represents the component of the acceleration resolved in the

i-j plane. The *k*-component may be assumed to be anything whatever without having any effect upon the spectrum produced. Let us imagine, therefore, that the whole motion of the electrons takes place in the one plane perpendicular to the line OO'.

Each term of this infinite series in (25) represents a purely circular motion, the part within the bracket, affected by an exponential factor $e^{-\nu t}$, which causes the amplitude of the motion to diminish with time finally to zero. Hence each term separately becomes zero after an infinite time, and so the whole acceleration vanishes, as it should. For then it is considered that the two electrons are located at the opposite ends of a common diameter in a fixed circular orbit, whence the sum of the accelerations of the two is evidently zero.

Upon integration of this equation with respect to the time the sum of the velocities of the two electrons is obtained, namely:

$$\frac{d\rho_2}{dt} + \frac{d\rho_1}{dt} = \frac{k_{\tau_2}}{2}\sum_{\tau=1}^{\tau=\infty}\left\{\nu^2 e^{-\nu t}\left[(-\sin\nu t - \cos\nu t)i + (-\cos\nu t + \sin\nu t)j\right]\right\} \quad . \quad . \quad . \quad . \quad (26)$$

No constant of integration is added because, after an infinite time, when the two electrons are in their final orbit, the sum of their velocities must evidently vanish.

A second integration gives us the sum of the two position vectors of the two electrons, namely:

$$\rho_2 + \rho_1 = \frac{k_{\tau_2}}{2}\sum_{\tau=1}^{\tau=\infty}\left\{\nu e^{-\nu t}\left[(\cos\nu t)i - (\sin\nu t)j\right]\right\} \quad . \quad (27)$$

No constant of integration is added here either, because, after an infinite time, when the two electrons are in their final orbit at opposite ends of a common diameter, evidently the sum of their position vectors must be zero.

It will be noticed that the factor ν^3, which was introduced into the equation for the acceleration (25), is reduced by the first integration to ν^2 in (26) and is reduced again by the second integration to ν in (27). The question why we need any factor ν in the equation (27) requires some explanation. That is, why could we not have had ν^0 in (27), ν^1 in (26), and ν^2 in (25)? Some such multiplying factor is required in (27) as we may see by making the time equal to zero. Without any factor of this kind each term of the infinite series becomes equal to unity, for the sine of zero vanishes, the cosine becomes unity, and the exponential becomes unity. The sum of the series therefore becomes the sum of an infinite number of units and is infinite and not finite. The presence of the factor ν, however, makes the series finite, as we shall see. There is, of course, no proof that this factor should be simply ν; but we cannot seek for proofs throughout this investigation until some particular example of it is completed so that comparisons may be made between the results of the assumptions and the known experimental facts.

Although this equation, giving the sum of the two position vectors, is of some interest, and we shall discuss it more fully later, yet we much prefer to know how this sum may be divided up into its two components, ρ_2 and ρ_1. So long as the sum is not changed, we are evidently at liberty to divide it up in a number of possible ways, but probably only one of these ways is admissible.

There is reason to think that the quantity

$$\mu = 2\pi K\left(\frac{1}{\tau^2} + \frac{1}{(\tau + \tau_2)^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (28)$$

plays some part in the individual motions of the electrons as well as in the difference of their position vectors

$\rho_2 - \rho_1$. This quantity μ is precisely the same as ν, except that the sign of the second term in (24) is positive instead of negative. It is also thought that the sum of μ and ν is involved in the individual motions of the electrons. We shall make the arbitrary assumption that the vector difference of the accelerations of the two electrons has the following expression:

$$\frac{d^2\rho_2}{dt^2} - \frac{d^2\rho_1}{dt^2} = k_{\tau_2}\sum_{\tau=1}^{\tau=\infty}\left\{\mu^3 e^{-\mu t}[(\sin \mu t)i + (\cos \mu t)j]\right\}$$
$$- B_{\tau_2}\sum_{\tau=1}^{\tau=\infty}\left\{(\mu+\nu)^3 e^{-(\mu+\nu)t}[(\sin(\mu+\nu)t)i\right.$$
$$\left.+ (\cos(\mu+\nu)t)j]\right\} - A(4\pi K)^2[(\cos 4\pi Kt)i$$
$$- (\sin 4\pi Kt)j]. \quad \ldots \ldots \ldots \ldots \quad (29)$$

The first summation in this is exactly analogous with equation (25), giving the sum of the accelerations, except that μ replaces ν. The second summation is an entirely analogous summation in which $\mu + \nu$ replaces the ν of equation (25), and a new constant, B_{τ_2}, replaces the k_{τ_2}. The last term is not a summation but a single term with a new constant A. This term may be regarded, if we please, merely as the first term of the preceding summation in which the exponent of the Naperian base e is zero, and so it does not appear. This last term is the only one which is left when the time is infinite, because both of the summations vanish, due to the exponential factors. The difference of the accelerations in the final orbit is of course not zero like their sum, but is represented by a purely circular motion, namely the last term in the equation.

Assuming this equation to represent the difference of the vector accelerations, we may find the difference of the velocities by a first integration as follows:

$$\frac{d\rho_2}{dt} - \frac{d\rho_1}{dt} = \frac{k_{\tau_2}}{2}\sum_{\tau=1}^{\tau=\infty}\Big\{\mu^2 e^{-\mu t}[(-\sin\mu t - \cos\mu t)i$$

$$+ (-\cos\mu t + \sin\mu t)j]\Big\}$$

$$-\frac{B_{\tau_2}}{2}\sum_{\tau=1}^{\tau=\infty}\Big\{(\mu+\nu)^2 e^{-(\mu+\nu)t}[(-\sin(\mu+\nu)t$$

$$-\cos(\mu+\nu)t)i + (-\cos(\mu+\nu)t + \sin(\mu+\nu)t)j]\Big\}$$

$$-A(4\pi K)[(\sin 4\pi Kt)i + (\cos 4\pi Kt)j] \quad \ldots\ldots \quad (30)$$

No constant of integration is added in this case either, because of the known final condition. The difference of the velocities must be double the velocity of each electron and be represented by a circular motion such as is given by the last term of this equation.

A second integration gives the difference of the position vectors as follows:

$$\rho_2 - \rho_1 = \frac{k_{\tau_2}}{2}\sum_{\tau=1}^{\tau=\infty}\Big\{\mu e^{-\mu t}[(\cos\mu t)i - (\sin\mu t)j]\Big\}$$

$$-\frac{B_{\tau_2}}{2}\sum_{\tau=1}^{\tau=\infty}\Big\{(\mu+\nu)e^{-(\mu+\nu)t}[(\cos(\mu+\nu)t)i - (\sin(\mu+\nu)t)j]\Big\}$$

$$+A[(\cos 4\pi Kt)i - (\sin 4\pi Kt)j] \quad \ldots\ldots\ldots \quad (31)$$

No constant of integration is required here either, because of the known final condition of the motion. The difference of the position vectors must be equal to the diameter of the orbit and be represented by a purely circular motion as is given by the last term of the equation.

The accelerations, velocities, and position vectors of the two electrons individually may now be obtained by the simple addition and subtraction of the above expressions for the sums and the differences giving the following equations.

$$\frac{d^2\rho}{dt^2} = \frac{k_{\tau_2}}{2}\sum_{\tau=1}^{\tau=\infty}\left\{\nu^3 e^{-\nu t}[(\sin \nu t)i + (\cos \nu t)j]\right\}$$

$$\pm \frac{k_{\tau_2}}{2}\sum_{\tau=1}^{\tau=\infty}\left\{\mu^3 e^{-\mu t}[(\sin \mu t)i + (\cos \mu t)j]\right\}$$

$$\mp \frac{B_{\tau_2}}{2}\sum_{\tau=1}^{\tau=\infty}\left\{(\mu+\nu)^3 e^{-(\mu+\nu)t}[(\sin(\mu+\nu)t)i\right.$$

$$\left. + (\cos(\mu+\nu)t)j]\right\}$$

$$\mp \frac{A}{2}(4\pi K)^2[(\cos 4\pi Kt)i - (\sin 4\pi Kt)j]. \quad . \quad . \quad . \quad (32)$$

$$\frac{d\rho}{dt} = \frac{k_{\tau_2}}{4}\sum_{\tau=1}^{\tau=\infty}\left\{\nu^2 e^{-\nu t}[(-\sin \nu t - \cos \nu t)i\right.$$

$$\left. + (-\cos \nu t + \sin \nu t)j]\right\}$$

$$\pm \frac{k_{\tau_2}}{4}\sum_{\tau=1}^{\tau=\infty}\left\{\mu^2 e^{-\mu t}[(-\sin \mu t - \cos \mu t)i\right.$$

$$\left. + (-\cos \mu t + \sin \mu t)j]\right\}$$

$$\mp \frac{B_{\tau_2}}{4}\sum_{\tau=1}^{\tau=\infty}\left\{(\mu+\nu)^2 e^{-(\mu+\nu)t}[(-\sin(\mu+\nu)t\right.$$

$$\left. - \cos(\mu+\nu)t)i + (-\cos(\mu+\nu)t + \sin(\mu+\nu)t)j]\right\}$$

$$\mp \frac{A}{2}(4\pi K)[(\sin 4\pi Kt)i + (\cos 4\pi Kt)j]. \quad . \quad . \quad . \quad (33)$$

$$\rho = \frac{k_{\tau_2}}{4}\sum_{\tau=1}^{\tau=\infty}\left\{\nu e^{-\nu t}[(\cos \nu t)i - (\sin \nu t)j]\right\}$$

$$\pm \frac{k_{\tau_2}}{4}\sum_{\tau=1}^{\tau=\infty}\left\{\mu e^{-\mu t}[(\cos \mu t)i - (\sin \mu t)j]\right\}$$

$$\mp \frac{B_{\tau_2}}{4}\sum_{\tau=1}^{\tau=\infty}\left\{(\mu+\nu)e^{-(\mu+\nu)t}[(\cos(\mu+\nu)t)i - (\sin(\mu+\nu)t)j]\right\}$$

$$\pm \frac{A}{2}[(\cos 4\pi Kt)i - (\sin 4\pi Kt)j]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

The upper signs should be used for the second and the lower signs for the first electron. The constants k_{τ_2}, B_{τ_2} and A are evidently connected with the initial and the final condition of the motion. In order to determine these constants let us write down the equations from (25) to (34), first assuming that the time is zero for the initial condition and then assuming that the time is infinity for the final condition. We have[1]

$$\left(\frac{d^2\rho_2}{dt^2} + \frac{d^2\rho_1}{dt^2}\right)_0 = k_{\tau_2}\Sigma(\nu^3)j \quad \ldots \ldots \quad (35)$$

$$\left(\frac{d\rho_2}{dt} + \frac{d\rho_1}{dt}\right)_0 = -\frac{k_{\tau_2}}{2}\Sigma(\nu^2)\,(i+j) \quad \ldots \ldots \quad (36)$$

$$(\rho_2 + \rho_1)_0 = \frac{k_{\tau_2}}{2}\Sigma(\nu)i \quad \ldots \ldots \quad (37)$$

$$\left(\frac{d^2\rho_2}{dt^2} - \frac{d^2\rho_1}{dt^2}\right)_0 = -A(4\pi K)^2 i + [k_{\tau_2}\Sigma(\mu^3) - B_{\tau_2}\Sigma(\mu+\nu)^3]j \quad \ldots \ldots \quad (38)$$

$$\left(\frac{d\rho_2}{dt} - \frac{d\rho_1}{dt}\right)_0 = \left[-\frac{k_{\tau_2}}{2}\Sigma(\mu^2) + \frac{B_{\tau_2}}{2}\Sigma(\mu+\nu)^2\right]i + \left[-\frac{k_{\tau_3}}{2}\Sigma(\mu^2) + \frac{B_{\tau_2}}{2}\Sigma(\mu+\nu)^2 - 4\pi KA\right]j \quad \ldots \ldots \quad (39)$$

$$(\rho_2 - \rho_1)_0 = \left[\frac{k_{\tau_2}}{2}\Sigma(\mu) - \frac{B_{\tau_2}}{2}\Sigma(\mu+\nu) + A\right]i \ldots \quad (40)$$

$$\left(\frac{d^2\rho}{dt^2}\right)_0 = \mp\frac{A}{2}(4\pi K)^2 i + \left[\pm\frac{k_{\tau_2}}{2}(\Sigma(\mu^3) \pm \Sigma(\nu^3)) \mp \frac{B_{\tau_2}}{2}\Sigma(\mu+\nu)^3\right]j \ldots \ldots \quad (41)$$

[1] The Σ's occur so frequently that it facilitates the printing to omit the limits $\tau = 1$ and $\tau = \infty$. Whenever these limits are not expressed in this volume, it will be understood that the limits intended are $\tau = 1$ to $\tau = \infty$.

$$\left(\frac{d\rho}{dt}\right)_0 = \left\{ \mp \frac{k_{\tau_2}}{4} [\Sigma(\mu^2) \pm \Sigma(\nu^2)] \pm \frac{B_{\tau_2}}{4} \Sigma(\mu + \nu)^2 \right\} i$$

$$+ \left\{ \mp \frac{k_{\tau_2}}{4} [\Sigma(\mu^2) \pm \Sigma(\nu^2)] \pm \frac{B_{\tau_2}}{4} \Sigma(\mu+\nu)^2 \mp 2\pi KA \right\} j \quad . \quad (42)$$

$$(\rho)_0 = \left[\pm \frac{k_{\tau_2}}{4} (\Sigma(\mu) \pm \Sigma(\nu)) \mp \frac{B_{\tau_2}}{4} \Sigma(\mu + \nu) \right.$$

$$\left. \pm \frac{A}{2} \right] i \quad . \; . \; . \; . \; . \; . \; . \quad (43)$$

$$\left(\frac{d^2\rho_2}{dt^2} + \frac{d^2\rho_1}{dt^2}\right)_\infty = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (44)$$

$$\left(\frac{d\rho_2}{dt} + \frac{d\rho_1}{dt}\right)_\infty = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (45)$$

$$(\rho_2 + \rho_1)_\infty = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (46)$$

$$\left(\frac{d^2\rho_2}{dt^2} - \frac{d^2\rho_1}{dt^2}\right)_\infty = -(4\pi K)^2 A[(\cos 4\pi Kt)i$$

$$- (\sin 4\pi Kt)j] \quad . \; . \; . \; . \; . \; . \; . \; . \quad (47)$$

$$\left(\frac{d\rho_2}{dt} - \frac{d\rho_1}{dt}\right)_\infty = -(4\pi K)A[(\sin 4\pi Kt)i$$

$$+ (\cos 4\pi Kt)j] \quad . \; . \; . \; . \; . \; . \; . \; . \quad (48)$$

$$(\rho_2 - \rho_1)_\infty = A[(\cos 4\pi Kt)i - (\sin 4\pi Kt)j] \quad . \; . \quad (49)$$

$$\left(\frac{d^2\rho}{dt^2}\right)_\infty = \mp (4\pi K)^2 \frac{A}{2} [(\cos 4\pi Kt)i$$

$$- (\sin 4\pi Kt)j] \quad . \; . \; . \; . \; . \; . \; . \; . \quad (50)$$

$$\left(\frac{d\rho}{dt}\right)_\infty = \mp (4\pi K) \frac{A}{2} [(\sin 4\pi Kt)i$$

$$+ (\cos 4\pi Kt)j] \quad . \; . \; . \; . \; . \; . \; . \; . \quad (51)$$

$$(\rho)_\infty = \pm \frac{A}{2} [(\cos 4\pi Kt)i - (\sin 4\pi Kt)j]. \quad . \quad (52)$$

VI

AS before, the upper signs apply to the motion of the second electron and the lower to that of the first. It appears from these equations that the final motion of the electrons is a simple circular motion at the opposite ends of a common diameter according to (52). Denoting the radius of this orbit by a, the constant A is therefore equal to twice the radius of the orbit, and we have

$$A = 2a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

This constant is the same for every value of τ_2, and the radius is independent of the initial conditions and of the amount of energy absorbed by the system. The frequency of revolution in the orbit is equal to $2K$, twice the Rydberg constant, namely $2 \times 3.290 \times 10^{15}$. This is supposed to come about because the A term in the difference-equation (29) is considered to belong to the $(\mu + \nu)$ series, being that term of it which has the exponential factor e^0, and in which $\tau = 1$. If we add μ and ν, which are given in (28) and (24), the second term in the parenthesis cancels, and we have for all values of τ_2

$$\mu + \nu = 4\pi K \frac{1}{\tau^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

When $\tau = 1$, this gives the angular velocity $4\pi K$ and the frequency $2K$.

Let us next consider the initial conditions when the time is zero. Equation (43) shows that both electrons are located upon the i-axis when the time is zero, there

being no j-component in the equation. In order to obtain values of the constants k_{T_2} and B_{T_2}, let it be supposed that when the time is zero the outermost electron has reached its maximum distance away from the nucleus and is about to return again. This is the moment when the absorption of energy ceases and the radiation of it begins. If this electron is not moving away from the nucleus, the only motion that it can have at this time is a motion in a direction perpendicular to its radius vector. Its radius vector is, as we have just mentioned, along the i-axis at this time. Hence its velocity must be along the j-axis when the time is zero. The i-component of the velocity in equation (42) may then be equated to zero. We shall regard the second electron as the more distant electron at this time and therefore use the upper signs in (42), which apply to this electron, giving

$$-\frac{k_{T_2}}{4}(\Sigma(\mu^2)+\Sigma(\nu^2))+\frac{B_{T_2}}{4}\Sigma(\mu+\nu)^2=0. \quad . \ (55)$$

Whence is derived a relation between k_{T_2} and B_{T_2} as follows:

$$\frac{k_{T_2}}{B_{T_2}}=\frac{\Sigma(\mu+\nu)^2}{\Sigma(\mu^2)+\Sigma(\nu^2)}. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (56)$$

The whole velocity of e_2 is then the j-component of (42), which reduces to the simple expression

$$\left(\frac{d\rho_2}{dt}\right)_0=-2\pi KAj=-(4\pi Ka)j. \quad . \ . \ . \ . \ . \ (57)$$

The sum of the velocities of e_2 and e_1 is given by (36). Subtracting from this the velocity of e_2 in (57), the velocity of e_1 is as follows:

$$\left(\frac{d\rho_1}{dt}\right)_0=-\frac{k_{T_2}}{2}\Sigma(\nu^2)[i+j]+(4\pi Ka)j. \quad . \ . \ . \ (58)$$

This equation does not admit of the i-component of the velocity of e_1 being zero, since neither k_{τ_2} nor $\Sigma(\nu^2)$ can vanish, and consequently does not admit of the velocity of e_1 being perpendicular to its radius vector, which, as we have seen, is also along the i-axis when the time is zero. It is to be supposed that there is no sudden or abrupt change in the direction of motion of the electrons at the time when the absorption of energy ceases and the radiation of it begins, namely at the zero time. The velocity of e_1 cannot be zero at this time, according to these equations, and it must have some direction. It is also natural to suppose that there is a symmetry between the outgoing and the return motions, and we may expect this symmetry to exist at the zero time. The only two directions that satisfy these conditions are either along the radius vector or perpendicular to it. Since the velocity of e_1 cannot be perpendicular to its radius vector, as we have seen, we shall take its direction of motion when the time is zero along the radius vector, namely along the i-axis, and shall equate the j-component of (58) to zero, giving

$$k_{\tau_2} = \frac{8\pi Ka}{\Sigma(\nu^2)}, \quad \ldots \ldots \ldots \ldots \quad (59)$$

and

$$\left(\frac{d\rho_1}{dt}\right)_0 = -\frac{k_{\tau_2}}{2}\Sigma(\nu^2)i = -(4\pi Ka)i. \quad . \; . \; . \quad (60)$$

Comparing (57) with (60) it appears that the initial velocities of the two electrons have the same value, that of e_2 being in the direction of $-j$ and of e_1 in the direction of $-i$. Moreover, comparing these values with the coefficient of (51), which expresses the velocity in the final orbit, it appears that the initial and final values of the velocities are the same. According to this result there

is no change whatever in the kinetic energy of the electrons between their initial and final motions.

By combining the value of the constant k_{τ_2} with the ratio of k_{τ_2} to B_{τ_2} in (56), the value of B_{τ_2} is as follows:

$$B_{\tau_2} = \frac{8\pi K a}{\Sigma(\mu + \nu)^2}\left(1 + \frac{\Sigma(\mu^2)}{\Sigma(\nu^2)}\right). \quad . \quad . \quad . \quad . \quad (61)$$

Having determined expressions for the three constants, k_{τ_2}, B_{τ_2} and A, which enter into the equations of motion from (25) on, we might now substitute them in these equations and calculate numerical examples of the motion for some fixed value of τ_2. The labor connected with this numerical calculation is, however, considerable even for a single example, and it will be deferred until a later section of the work (See Chapter XII).

VII

THE sums of the various powers of μ and ν and $(\mu + \nu)$ enter into these equations through the constants k_{τ_2} and B_{τ_2}, and we shall next give some consideration to the determination of their numerical values, which will be required before further progress can be made in the interpretation of the theory above given. Consider first the summation of $(\mu + \nu)$ as given in (54) above, when τ takes all values from 1 to infinity, and denote the sum by s_1. We have

$$s_1 = \Sigma(\mu+\nu) = \Sigma\mu + \Sigma\nu = 4\pi K \Sigma \frac{1}{\tau^2} = 4\pi K(1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} \cdots) \quad (62)$$

The sum of the series of numbers in the parenthesis is known to be exactly equal to $\pi^2/6$. Hence, we have

$$s_1 = 4\pi K \frac{\pi^2}{6} = 4\pi K \times 1.644{,}934{,}066{,}8. \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

In a similar manner, denoting by s_2 the sum of $(\mu + \nu)^2$, and by s_3 the sum of $(\mu + \nu)^3$, we have

$$s_2 = \Sigma(\mu+\nu)^2 = (4\pi K)^2 \Sigma \frac{1}{\tau^4} = (4\pi K)^2 \left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} \cdots\right), \ . \ (64)$$

and

$$s_3 = \Sigma(\mu+\nu)^2 = (4\pi K)^3 \Sigma \frac{1}{\tau^6} = (4\pi K)^3 \left(1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} \cdots\right) \cdot \quad (65)$$

The numerical values of the sums of the series

$$\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

have been calculated. A table of these sums to fifteen decimal places and for values of n from 1 to 35 is given on page 554 of De Morgan's calculus. This table is reproduced here to nine places of decimals and for the even values of n only in (67). By its means we find

n	$\sum_1^\infty \frac{1}{\tau^n}$
2	1.644,934,067
4	1.082,323,234
6	1.017,343,062
8	1.004,077,356
10	1.000,994,575
12	1.000,246,087
14	1.000,061,248
16	1.000,015,282
18	1.000,003,817
20	1.000,000,954
22	1.000,000,238,45
24	1.000,000,059,61
26	1.000,000,014,90
28	1.000,000,003,73
30	1.000,000,000,93
32	1.000,000,000,23
34	1.000,000,000,06
36	1.000,000,000,01

. . . (67)

$$s_2 = (4\pi K)^2 \times 1.082{,}323{,}233{,}7 \ldots \quad (68)$$

$$s_3 = (4\pi K)^3 \times 1.017{,}343{,}062. \quad (69)$$

Let us next consider the numerical values of the sum of ν and of μ. We have

$$\Sigma\nu = 2\pi K \Sigma\left(\frac{1}{\tau^2} - \frac{1}{(\tau + \tau_2)^2}\right). \quad (70)$$

If $\tau_2 = 3$ say, then

$$\Sigma\nu = 2\pi K\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots \right.$$
$$\left. - \frac{1}{4^2} - \frac{1}{5^2} - \frac{1}{6^2} - \cdots\right) = 2\pi K\left(1 + \frac{1}{2^2} + \frac{1}{3^2}\right). \quad (71)$$

Hence, we may say in general that

$$\Sigma\nu = 2\pi K \sum_{1}^{\tau_2} \frac{1}{\tau^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

Similarly, we have the value of

$$\Sigma\mu = 2\pi K \Sigma\left(\frac{1}{\tau^2} + \frac{1}{(\tau+\tau_2)^2}\right), \quad . \quad . \quad . \quad . \quad . \quad (73)$$

and, if $\tau_2 = 3$ say, we have

$$\Sigma\mu = 2\pi K\left(\frac{2}{1^2} + \frac{2}{2^2} + \frac{2}{3^2} + \frac{2}{4^2} + \frac{2}{5^2} + \ldots - \frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2}\right)$$

$$= 2\pi K\left(\frac{\pi^2}{3} - \frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2}\right). \quad . \quad . \quad . \quad . \quad . \quad (74)$$

Hence, generally

$$\Sigma\mu = 2\pi K\left(\frac{\pi^2}{3} - \sum_{1}^{\tau_2} \frac{1}{\tau^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

It is seen that we obtain (63) by adding (72) and (75). The following table (76*a*) gives the values of $1/\tau$, $1/\tau^2$ and $1/\tau^4$ for the first ten values of τ. And the table (77*a*) gives the sums of $1/\tau$, $1/\tau^2$ and $1/\tau^4$ for the first ten values of τ.

τ	$\frac{1}{\tau}$	$\frac{1}{\tau^2}$	$\frac{1}{\tau^4}$
1	1.000,000,000	1.000,000,000	1.000,000,000
2	0.500,000,000	0.250,000,000	0.062,500,000
3	0.333,333,333	0.111,111,111	0.012,345,679
4	0.250,000,000	0.062,500,000	0.003,906,250
5	0.200,000,000	0.040,000,000	0.001,600,000
6	0.166,666,667	0.027,777,778	0.000,771,605
7	0.142,857,143	0.020,408,163	0.000,416,495
8	0.125,000,000	0.015,625,000	0.000,244,141
9	0.111,111,111	0.012,345,679	0.000,152,416
10	0.100,000,000	0.010,000,000	0.000,100,000
Sum	2.928,968,3	1.549,767,73	1.082,036,586

. . (76a)

τ	$\sum_1^{\tau_2} \frac{1}{\tau}$	$\sum_1^{\tau_2} \frac{1}{\tau^2} = z$	$\sum_1^{\tau_2} \frac{1}{\tau^4}$
1	1.000,000	1.000,000	1.000,000,000
2	1.500,000	1.250,000	1.062,500,000
3	1.833,333	1.361,111	1.074,845,679
4	2.083,333	1.423,611	1.078,751,929
5	2.283,333	1.463,611	1.080,351,929
6	2.445,000	1.491,388	1.081,123,534
7	2.592,857	1.511,797	1.081,540,029
8	2.717,857	1.527,422	1.081,784,170
9	2.828,968	1.539,768	1.081,936,586
10	2.928,968	1.549,768	1.082,036,586
..	....	....	....
∞	∞	$1.644,934 = \frac{\pi^2}{6}$	1.082,323,233,7

. . (77a)

Let us next determine the values of $\Sigma\nu^2$ and $\Sigma\mu^2$ in terms of sums of the powers of $1/\tau^n$ so that the table of numerical values of $1/\tau^n$ in (67) may be used for calculating $\Sigma\nu^2$ and $\Sigma\mu^2$. We have

$$\Sigma\nu^2 = (2\pi K)^2 \Sigma\left(\frac{1}{\tau^2} - \frac{1}{(\tau+\tau_2)^2}\right)^2$$

$$= (2\pi K)^2\left[\Sigma\frac{1}{\tau^4} + \Sigma\frac{1}{(\tau+\tau_2)^4} - 2\Sigma\frac{1}{\tau^2(\tau+\tau_2)^2}\right]. \quad (76)$$

$\Sigma\frac{1}{\tau^4}$ is given in the table (67) as 1.082,323. If $\tau_2 = 3$ say, then

$$\Sigma\frac{1}{(\tau+3)^4} = \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \cdots = 1.082,323 - \sum_1^3 \frac{1}{\tau^4} \quad (77)$$

and, generally,

$$\Sigma\frac{1}{(\tau+\tau_2)^4} = 1.082,323 - \sum_1^{\tau_2}\frac{1}{\tau^4} \quad . \quad . \quad . \quad . \quad (78)$$

This leaves the last term of (76) to be evaluated, which may be effected by resolving $1/\tau^2(\tau + \tau_2)^2$ into four partial fractions according to the following identity:

$$\frac{1}{\tau^2(\tau+\tau_2)^2} = \frac{1}{\tau_2{}^2\tau^2} - \frac{2}{\tau_2{}^3\tau} + \frac{1}{\tau_2{}^2(\tau+\tau_2)^2} + \frac{2}{\tau_2{}^3(\tau+\tau_2)} \quad (79)$$

In summing this expression term by term, τ_2 is to be regarded as constant. Since we know that

$$\Sigma\frac{1}{\tau^2} = \frac{\pi^2}{6} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

and

$$\Sigma\frac{1}{(\tau+\tau_2)^2} = \frac{\pi^2}{6} - \sum_1^{\tau_2}\frac{1}{\tau^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

it follows that the sums of the first and the third terms of (79) give

$$\frac{1}{\tau_2{}^2}\left(\frac{\pi^2}{3} - \sum_1^{\tau_2}\frac{1}{\tau^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

Similarly, since

$$\Sigma\frac{1}{\tau} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

and, when $\tau_2 = 3$ say,

$$\Sigma\frac{1}{\tau+\tau_2} = \frac{1}{4} + \frac{1}{5} + \ldots, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

we find generally that the sum of the second and fourth terms in (79) give

$$-\frac{2}{\tau_2{}^3}\sum_1^{\tau_2}\frac{1}{\tau}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

By adding (85) and (82) we obtain the complete sum of the last term of (76). The whole sum of (76) is, therefore,

$$\Sigma\nu^2 = (2\pi K)^2\left\{2 \times 1.082{,}323 - \frac{2}{3}\frac{\pi^2}{\tau_2{}^2} + \frac{4}{\tau_2{}^3}\sum_1^{\tau_2}\frac{1}{\tau} + \frac{2}{\tau_2{}^2}\sum_1^{\tau_2}\frac{1}{\tau^2} - \sum_1^{\tau_2}\frac{1}{\tau^4}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

To obtain the sum of μ^2 simply change the negative sign in the last term of (76) to positive, giving

$$\Sigma\mu^2 = (2\pi K)^2 \left\{ 2 \times 1.082,323 + \frac{2}{3}\frac{\pi^2}{\tau_2^2} - \frac{4}{\tau_2^3}\sum_1^{\tau_2}\frac{1}{\tau} - \frac{2}{\tau_2^2}\sum_1^{\tau_2}\frac{1}{\tau^2} - \sum_1^{\tau_2}\frac{1}{\tau^4} \right\} \quad (87)$$

By the use of these formulæ and the table (67), the following table, (88), of values of $\Sigma\nu^2$ and $\Sigma\mu^2$ have been calculated. Let us denote by x, y, and z the following expressions, which are tabulated in (88) and (77*a*).

$$x = \frac{\Sigma\nu^2}{(2\pi K)^2}, \quad (89)$$

$$y = \frac{\Sigma\mu^2}{(2\pi K)^2}, \quad (90)$$

$$z = \sum_1^{\tau_2}\frac{1}{\tau^2}. \quad (91)$$

τ_2	$x = \frac{\Sigma\nu^2}{(2\pi K)^2}$	$y = \frac{\Sigma\mu^2}{(2\pi K)^2}$	$\frac{\Sigma(\mu\nu)}{(2\pi K)^2}$
1	0.584,910	1.744,382	1.000,000
2	0.832,212	1.372,080	1.062,500
3	0.932,793	1.246,808	1.074,845,68
4	0.982,820	1.188,968	1.078,751,93
5	1.011,261	1.157,328	1.080,351,93
6	1.028,885	1.138,161	1.081,123,534
7	1.040,769	1.125,443	1.081,540,029
8	1.049,019	1.116,705	1.081,784,170
9	1.055,020	1.110,400	1.081,936,586
10	1.059,524	1.105,686	1.082,039,586
..	....	....	....
∞	1.082,323,233,7	1.082,323,233,7	1.082,323,233,7

(88)

These are plotted as curves in Fig. 2. The value of the constant k_{τ_2}, as determined above in (59), is, therefore,

$$k_{\tau_2} = \frac{2a}{\pi K}\frac{1}{x}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

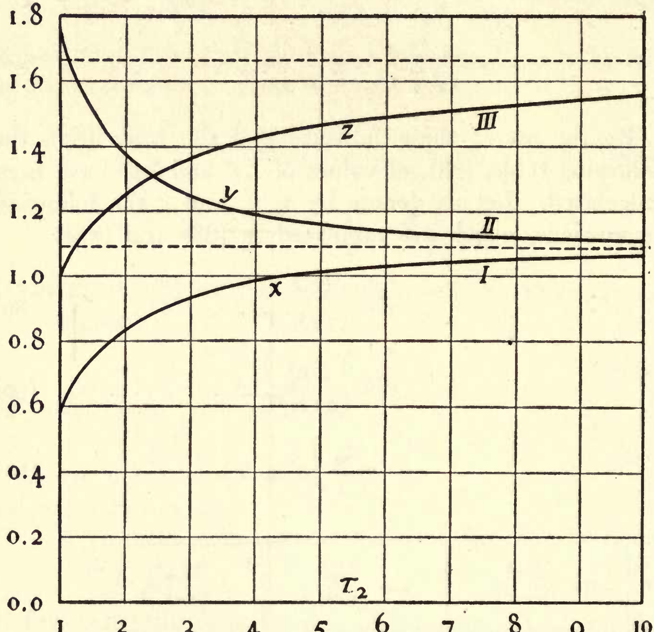

Fig. 2.

which is proportional to the reciprocal of the function x shown in Fig. 2. The constant B_{τ_2}, according to (61) and (68), is

$$B_{\tau_2} = \frac{a}{2\pi K \times 1.082,323}\left(1 + \frac{y}{x}\right), \quad . \quad . \quad . \quad . \quad (93)$$

and is proportional to the function $1 + y/x$. The functions $1/x$, y/x, $1 + y/x$ and z/x are given in Table (94), and shown as curves in Fig. 3.

τ_2	$\frac{1}{x}$	$\frac{y}{x}$	$1+\frac{y}{x}$	$\frac{z}{x}$
1	1.709,664,7	2.982,308,39	3.982,308,39	1.709,664,7
2	1.201,616,9	1.648,714,51	2.648,714,51	1.502,021,1
3	1.072,049,2	1.336,639,53	2.336,639,53	1.459,178,1
4	1.017,480,3	1.209,751,53	2.209,751,53	1.448,496,3
5	0.988,864,3	1.144,440,34	2.144,440,34	1.447,312,7
6	0.971,025,9	1.106,208,15	2.106,208,15	1.449,518,6
7	0.960,828,0	1.081,357,15	2.081,357,15	1.452,576,9
8	0.953,271,6	1.064,523,16	2.064,523,16	1.456,048,0
9	0.947,849,3	1.052,491,86	2.052,491,86	1.459,468,0
10	0.943,820,0	1.043,568,56	2.043,568,56	1.462,702,0
..	....	....	....	....
∞	0.923,938,4	1.000,000,00	2.000,000,00	1.519,818

(94)

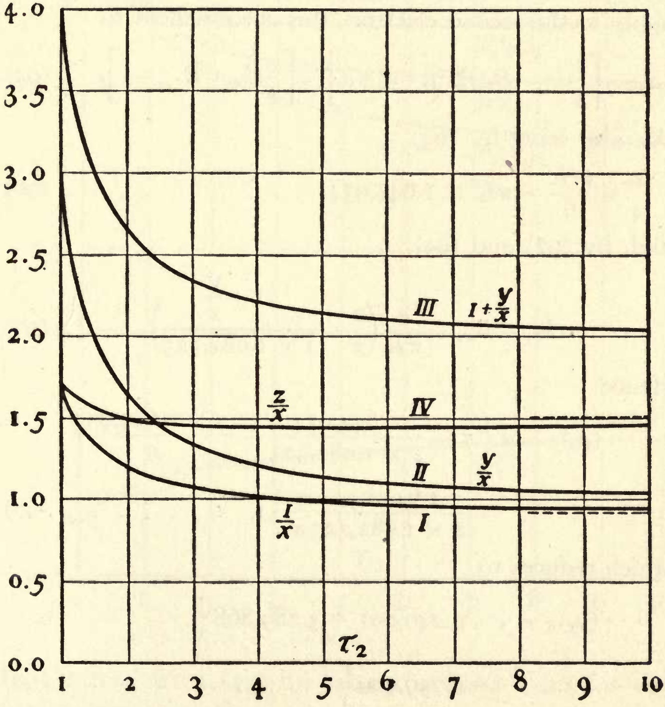

Fig. 3.

VIII

LET us next return to the initial conditions of the two electrons expressed by the equations from (35) to (43), for which purpose the numerical values of the summations of ν and μ have been determined. Considering the initial position vector $(\rho)_0$ in (43), using the upper signs which apply to the second electron, this is equivalent to

$$(\rho_2)_0 = \left[\frac{1}{4}(k_{T_2} - B_{T_2})\Sigma(\mu+\nu) + a\right]i = \left[\frac{s_1}{4}(k_{T_2} - B_{T_2}) + a\right]i \quad . \quad (95)$$

We also have by (63)

$$\frac{s_1}{4} = \frac{\pi^3 K}{6} = \pi K \times 1.644{,}934, \quad \dots\dots\dots \quad (96)$$

and, by (92) and (93),

$$k_{T_2} - B_{T_2} = \frac{a}{\pi K}\left(\frac{2}{x} - \frac{1 + \dfrac{y}{x}}{2 \times 1.082{,}323}\right). \quad . \quad . \quad . \quad (97)$$

Hence

$$(\rho_2)_0 = a\left\{1 - \frac{1.644{,}934}{2 \times 1.082{,}323} + \frac{2 \times 1.644{,}934}{x} - \frac{1.644{,}934}{2 \times 1.082{,}323}\frac{y}{x}\right\}i, \quad \dots\dots \quad (98)$$

which reduces to

$$(\rho_2)_0 = a\left\{0.240{,}091 + 3.289{,}868\frac{1}{x} - 0.759{,}909\frac{y}{x}\right\}i \quad \dots\dots\dots \quad (99)$$

Again, we have the sum of the position vectors initially in (37),

$$(\rho_2 + \rho_1)_0 = \frac{1}{2} k_{\tau_2} \Sigma(\nu) i = \frac{a}{\pi K} \frac{1}{x} \Sigma(\nu) i = 2a \frac{z}{x} i. \quad . \quad . \quad (100)$$

Hence, the value of $(\rho_1)_0$ for the first electron is obtained by subtracting (99) from (100), or the same value might

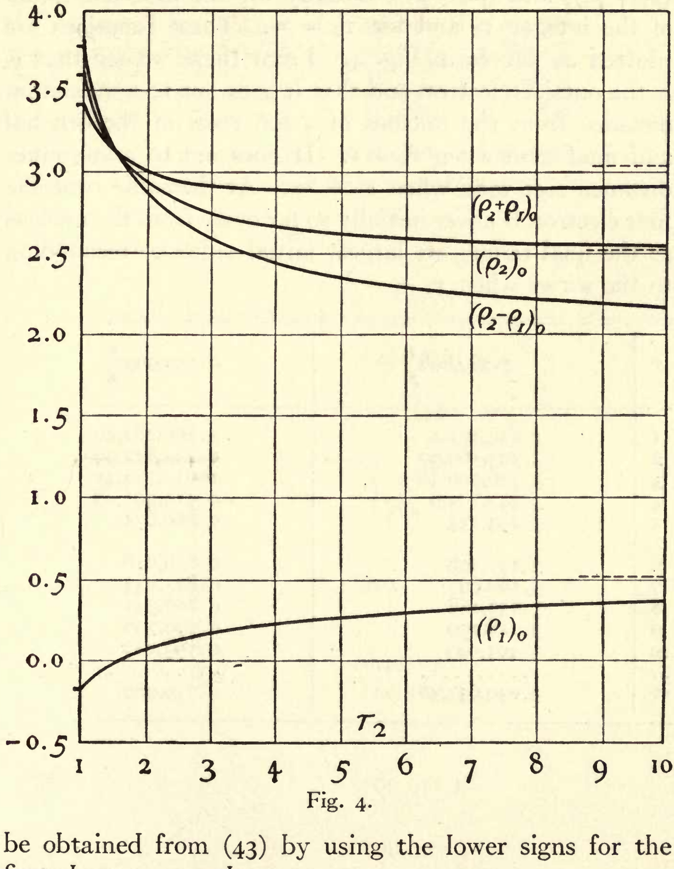

Fig. 4.

be obtained from (43) by using the lower signs for the first electron, namely

$$(\rho_1)_0 = a\left\{-0.240{,}091 - 3.289{,}868\frac{1}{x} + 0.759{,}909\frac{y}{x} + \frac{2z}{x}\right\}i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (101)$$

The following table (102) gives the values of $(\rho_2)_0$, $(\rho_1)_0$, $(\rho_2 + \rho_1)_0$ and $(\rho_2 - \rho_1)_0$ initially for the first ten values of the integer τ_2 and for $\tau_2 = \infty$. These functions are plotted as curves in Fig. 4. From these we see that e_2 is the outside electron and that it goes out to a maximum distance from the nuclues of 3.598 radii of the original and final orbit when $\tau_2 = 1$. It goes out to a minimum distance 2.52 radii when $\tau_2 = \infty$. At the same time the first electron is never initially so far away from the nucleus as the final radius, its largest initial value corresponding to the series when $\tau_2 = \infty$.

τ_2	$3.289{,}868\frac{1}{x}$	$0.759{,}909\frac{y}{x}$
1	5.624,571,2	2.266,282,99
2	3.593,160,99	1.252,872,99
3	3.526,900,36	1.015,724,41
4	3.347,375,9	0.919,301,08
5	3.253,233	0.869,671
6	3.197,508	0.840,618
7	3.160,997	0.821,733
8	3.136,138	0.808,941
9	3.118,299	0.799,798
10	3.105,043	0.793,017
..	....	....
∞	3.039,635,38	0.759,909

τ_2	$(\rho_2)_0$	$(\rho_2 + \rho_1)_0$	$(\rho_1)_0$	$(\rho_2 - \rho_1)_0$
1	3.598,379,2	3.419,329,4	−0.179,149,8	3.777,429,0
2	2.940,379	3.004,042,2	0.063,663,2	2.876,715,8
3	2.751,266,6	2.918,356,2	0.167,089,6	2.584,177,0
4	2.668,156,8	2.896,992,6	0.228,826,8	2.439,339,0
5	2.623,654	2.894,625	0.270,972	2.352,682
6	2.596,981	2.899,037	0.302,056	2.294,926
7	2.579,355	2.905,154	0.325,799	2.253,557
8	2.567,288	2.912,096	0.344,808	2.222,480
9	2.558,592	2.918,936	0.360,344	2.198,248
10	2.552,117	2.925,404	0.373,287	2.178,830
..	....	....	....	....
∞	2.519,818	3.039,636	0.519,818	2.000,000,0

. (102)

IX

LET us next give some consideration to the energy of the system composed of the hydrogen nucleus of charge $+ 2e$ and two negative electrons. The system cannot be regarded as a conservative system because it is capable of absorbing and parting with energy from and to its external surroundings. If the system were a conservative one, it has been shown by Helmholtz in his paper "On the conservation of force" in 1847 that the mutual forces between any two material points must be in the line joining them, and be a function of the distance between them. The converse proposition is also true. If two material points act on each other with a force depending as regards magnitude on their mutual distance, but not in the direction of the line joining them, they would be capable of producing in each other an increasing velocity and thus of generating or of dissipating energy.

The present electromagnetic theory as to the forces between two electrical charges in motion makes their directions not in the line joining their centers, and, accepting this as the fact, it follows that the system we are now considering is not a conservative one, but one capable of absorbing and of radiating energy. The fact that these hydrogen atoms are known to radiate and to absorb energy to and from external surroundings affords a confirmation of the electromagnetic theory in respect to the nature of the forces between two moving electrons, namely that the forces do not at all times act in the line joining their centers.

Let us denote the total energy content of the system, say at the time zero, by E_0. This energy may be considered to be the sum of its kinetic energy, T_0, and a potential energy, V_0, and we may write

$$T_0 + V_0 = E_0. \quad \ldots \ldots \quad (103)$$

At some later time, t, assuming that energy is being diminished by radiation and lost to the system by an amount, say R_t, since the time zero, the equation of energy at this future time, t, is then

$$T_t + V_t = E_0 - R_t, \quad \ldots \ldots \quad (104)$$

and, by subtraction of (104) from (103), the constant, E_0, disappears, giving

$$(T_0 - T_t) + (V_0 - V_t) = R_t. \quad \ldots \ldots \quad (105)$$

If the time, t, is taken as infinite, meaning by this that the two electrons have settled down in their final orbit and have ceased to radiate energy, the equations which have been discussed above tell us that the kinetic energy, T_0, is equal to the final kinetic energy, T_∞. Writing the energy equation (105) for the time $t = \infty$, we then have

$$V_0 - V_\infty = R_\infty. \quad \ldots \ldots \quad (106)$$

Now the energy radiated during any one excursion, or rather during the return from this excursion of the two electrons, will be different for each series, depending upon the particular value of τ_2. But, since the electrons come to the same final orbit every time, it is to be supposed that the potential, V_∞, is a constant quantity for all series. The initial potential energy, V_0, is, therefore, equal to the energy radiated to within a constant, or the difference between the initial potential energy and the energy radiated is constant for all series.

It now seems entirely legitimate to apply the Einstein

equation for the total energy radiated to the problem. This equation may be expressed as follows:

$$\text{Energy radiated} = h\nu' = \frac{h}{2\pi}\nu, \quad . \quad . \quad . \quad . \quad (107)$$

where ν' denotes the frequency and ν the angular velocity of each component frequency emitted by the system during the return of the electrons after displacement. According to this theory these frequencies are infinite in number corresponding to values of τ from 1 to infinity, τ_2 being a fixed integer for any one excursion. The total energy radiated in one operation of the system is, therefore,

$$R_\infty = \frac{h}{2\pi}\Sigma\nu = hK\sum_{1}^{\tau_2}\frac{1}{\tau^2} = hKz. \quad \text{(See (72).)} \quad . \quad . \quad (108)$$

This energy is evidently a function of the initial position vectors of the two electrons, for we have in (100)

$$(\rho_2 + \rho_1)_0 = 2a\frac{z}{x}i, \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

and, denoting the scalar values of ρ_2 and ρ_1 by r_2 and r_1, we obtain the ratio of R_∞ to $r_2 + r_1$ as follows:

$$\frac{R_\infty}{r_2 + r_1} = \frac{hK}{2a}x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (110)$$

That is to say, this ratio is proportional to the function, x, given in Table (88) and plotted in Fig. 2. It follows from this that the initial potential, V_0, is also a function of the initial sum of the distances because we have shown above that it differs from R_∞ by a constant quantity. We now have the equation, according to (106) and (108),

$$V_0 - V_\infty = R_\infty = \frac{h}{2\pi}\Sigma\nu \quad . \quad . \quad . \quad . \quad . \quad (111)$$

or

$$V_0 - \frac{h}{2\pi}\Sigma\nu = V_\infty, \text{ a constant.} \quad . \quad . \quad . \quad (112)$$

Let us next seek for some expression for V_0 which will reduce the difference, $V_0 - R_\infty$, to a constant quantity, the $R_\infty = \frac{h}{2\pi}\Sigma\nu$ being an infinite series of terms. Evidently the value

$$V_0 = -\frac{h}{2\pi}\Sigma\mu = -hK\left(\frac{\pi^2}{3} - z\right), \text{ (See (75).)}. \quad (113)$$

will accomplish this, for we have

$$-\frac{h}{2\pi}(\Sigma\mu + \Sigma\nu) = -\frac{h}{2\pi}s_1 = -hK\frac{\pi^2}{3}$$
$$= V_\infty. \text{ (See (63).)} \quad \ldots\ldots \quad (114)$$

Of course, an assumed value for V_0 which differs from (113) by a constant would also satisfy the condition in (112), but it is apparent that the principal part of V_0 must contain a series similar to $\Sigma\mu$, which will reduce the difference between V_0 and $\frac{h}{2\pi}\Sigma\nu$ to a constant. There is no proof at present that no constant should be included, except that the numerical values obtained by adhering to the very simple expression for V_0, $-\frac{h}{2\pi}\Sigma\mu$, which is analogous with the value of $R_\infty = \frac{h}{2\pi}\Sigma\nu$, gives a result in remarkable agreement with the experimental voltages of ionization for hydrogen, as will presently be shown.

The energy, V_∞, required to separate the two electrons completely away from the nucleus if they start from their original or final orbit is, by (114),

$$V_\infty = -hK\frac{\pi^2}{3} = -6.547 \times 10^{-27} \times 3.290 \times 10^{15} \times 3.289{,}868$$
$$= -.2154 \times 10^{-10}\frac{\pi^2}{3}$$
$$= -.7086 \times 10^{-10} \text{ ergs}, \quad \ldots\ldots \quad (115)$$

and the energy per electron is one half of this or $-.3543 \times 10^{-10}$ ergs. The energy required to separate the two electrons at zero time when the electron, e_2, is already in its position of maximum distance from the nucleus is given by V_0 in (113). The following Table (116) gives the values of V_0 for the first ten values of τ_2. This energy grows smaller and smaller with increasing values of τ_2. When $\tau_2 = 1$, we have

$$V_0 = -bK\left(\frac{\pi^2}{3} - 1\right)$$

$$= -0.4932 \times 10^{-10} \text{ ergs} \quad . \quad . \quad . \quad . \quad (117)$$

When $\tau_2 = 2$, $V_0 = -bK\left(\frac{\pi^2}{3} - 1.25\right)$

$$= -0.4394 \times 10^{-10} \text{ ergs} \quad . \quad . \quad . \quad . \quad (118)$$

When $\tau_2 = \infty$, $V_0 = -bK\left(\frac{\pi^2}{3} - \frac{\pi^2}{6}\right)$

$$= -0.3543 \times 10^{-10} \text{ ergs} \quad . \quad . \quad . \quad . \quad (119)$$

τ_2	$V_0 = -bK\left(\frac{\pi^2}{3} - z\right) = -0.2154 \times 10^{-10}(3.289{,}868 - z)$
1	$-.2154 \times 10^{-10} \times 2.289{,}868 = -0.4932 \times 10^{-10}$ ergs
2	" $\times 2.039{,}868 = -0.4394$ " "
3	" $\times 1.928{,}757 = -0.4155$ " "
4	" $\times 1.866{,}257 = -0.4020$ " "
5	" $\times 1.826{,}257 = -0.3934$ " "
6	" $\times 1.798{,}480 = -0.3874$ " "
7	" $\times 1.778{,}071 = -0.3830$ " "
8	" $\times 1.762{,}446 = -0.3796$ " "
9	" $\times 1.750{,}100 = -0.3770$ " "
10	" $\times 1.740{,}100 = -0.3748$ " "
..	....
..	....
∞	" $\times 1.644{,}934 = -0.3543$ " " . (116)

The energy required per electron is one half of these values. It is customary to express energy as the product of electromotive force and charge for the reason that it is

electromotive force that is observed by those who have conducted experiments to determine the critical values at which ionization of the gas takes place. When an electron of charge, e, is driven by an electromotive force, E, the energy, T, required is

$$T = eE, \quad \text{and} \quad E = \frac{T}{e} \quad . \quad . \quad . \quad . \quad . \quad (120)$$

If absolute units are used, ergs for energy, and absolute electrostatic units for charge, we obtain the electromotive force in absolute electrostatic units and not in volts. To convert into the practical units of electromotive force, volts, multiply by $c \times 10^{-8}$, giving

$$E = \frac{Tc}{e} \times 10^{-8} \text{ volts}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (121)$$

where T is expressed in ergs, and $e = 4.774 \times 10^{-10}$ electrostatic units. Hence

$$E = 0.6284 \times 10^{12} T \text{ volts}. \quad . \quad . \quad . \quad . \quad . \quad (122)$$

By means of this formula we may find the voltage required to impart to a single electron the energy T ergs. Substituting for T the energy required per electron to separate it from the atom, we obtain the following table

τ_2	E = ionizing voltage
1	15.496
2	13.806
3	13.055
4	12.631
5	12.361
6	12.172
7	12.034
8	11.927
9	11.845
10	11.776
..	......
..	......
∞	11.132

. . (123)

(123) of ionizing voltages, which may be compared with the experimental values.

It is considered that this theoretical result is in remarkable agreement with the recent experimental determination of the ionizing voltages for hydrogen by Davis and Goucher and others. These experimental results show that nothing whatever begins to happen in hydrogen until 11 volts is passed, and at a value just a little above 11 volts ionization sets in. According to the Table (123) ionization begins at 11.13 volts, corresponding to a value of τ_2 equal to ∞. According to the theory the ionization should be almost continuous from 11.13 volts up until we come to the very small values of τ_2, when the voltage takes larger jumps, ending with a maximum value of 15.5 volts. Davis and Goucher have observed a new type of ionization setting in at about 15.8 volts, which is only .3 of a volt greater than the theoretical maximum in the table. This small fraction of a volt is probably within the error of experimental measurement. Another type of ionization has been observed at 13.6 volts, which corresponds very closely with the second value in the table corresponding to $\tau_2 = 2$. All values corresponding to higher values of τ_2 are too near together to show as critical points in any of the experimental curves since they are merged into one another.

It is of interest to contrast this result with the values of the ionizing voltage for hydrogen derived from the Bohr theory by the experimenters above referred to. By the use of the formula

$$T = h\nu' = hK\left(\frac{1}{\tau_2^2} - \frac{1}{\tau_1^2}\right), \quad \ldots\ldots (124)$$

it is possible to derive, by putting $\tau_2 = 1$, and $\tau_1 = \infty$,

$$T = hK, \quad \ldots\ldots\ldots\ldots\ldots (125)$$

and by putting $\tau_2 = 1$, and $\tau_1 = 2$, to derive

$$T = \frac{3}{4} hK. \quad \ldots \ldots \ldots \ldots \quad (126)$$

The hK in (125) converted into volts gives

$$E = \frac{hKc}{e} \times 10^{-8} = 13.54 \text{ volts}, \quad \ldots \quad (127)$$

and the value in (126) is three quarters of this, namely

$$E = 10.15 \text{ volts}. \quad \ldots \ldots \ldots \quad (128)$$

The former value 13.54 is the largest value of the voltage given by this formula, and the 10.15 is the smallest value obtainable when $\tau_2 = 1$. If τ_2 were greater than unity, it is possible to obtain as small values as we please way down to zero.

The 10.15 volts is too small to represent any experimental ionizing voltage by nearly one volt, since nothing whatever happens until 11 volts are passed. The 13.54 falls near to the observed value of another type of ionizing voltage, but the 13.54 is the maximum possible value that this formula yields for any values of τ whatever, and there is no indication of any higher voltage of ionization in hydrogen given by the Bohr theory.

It is difficult to see why the voltages calculated in this way from the formula (124) should represent ionizing voltages at all, unless we always put $\tau_1 = \infty$.

X

UP to this point nothing has been learned concerning the absolute value of the kinetic energy, T_0, nor T_∞, which is equal to it, since they canceled each other and dropped out from the energy equation (105). Neither have we obtained any absolute value for the final radius of the orbit, a, in centimeters. The Bohr theory made use of the following very general theorem, upon which its most important results are based: "In every system consisting of electrons and positive nuclei, in which the nuclei are at rest and the electrons move in circular orbits with a velocity small compared with the velocity of light, the kinetic energy will be numerically equal to half the potential energy." If we should make use of this theorem, the kinetic energy of the two electrons in this system would be equal to one half of the potential energy in (114), $hK\frac{\pi^2}{6}\cdot$ The theorem quoted, however, assumes the inverse square law of force between electrons, which we have not assumed and do not propose to assume. Another result from the same theorem applied to the several stationary orbits in the Bohr theory was that the energy radiated is equal to the change in the kinetic energy, or that the sum of the energy radiated in coming from an infinite distance to the nucleus and the final kinetic energy is equal to the final potential energy. In the theory expressed by the equations given above, there is no change in the kinetic energy between the outer-

most position of the electron, e_2, and its final position. This is in direct contradiction to the theorem quoted.

It will be noticed that the energy radiated as expressed in (108) depends upon the value of τ_2, and that it varies between the values hK, when $\tau_2 = 1$, and $hK\frac{\pi^2}{6}$ when $\tau_2 = \infty$. The greater of these two values is just equal to half the final potential energy, as above shown, but it corresponds to the series where $\tau_2 = \infty$. The energy to separate the two electrons from the nucleus in this series is a minimum (see (119)), while the energy to separate them in the series where $\tau_2 = 1$ is a maximum (see (117)). We shall, therefore, regard the first series, where $\tau_2 = 1$, as the fundamental series, and shall consider that the kinetic energy of the two electrons in their final orbit is equal to the energy radiated in this series rather than in the head series, where $\tau_2 = \infty$. This energy is simply hK. The justification of this is to be found in the fact that it leads to the formula for the velocity of electrons in rings in complete agreement with a result previously obtained, and from which a correct numerical value of the Newtonian gravitational constant has been derived, as will be shown in a subsequent section. Let us, therefore, equate the kinetic energy of the two electrons in their final orbit to hK, giving

$$hK = 2\tfrac{1}{2}mv^2 = mv^2. \qquad (129)$$

From this is derived an expression for the velocity of the electrons in the ring, for, in any circular orbit where a point revolves with a frequency, n, the linear velocity is

$$v = 2\pi an. \qquad (130)$$

But the frequency in the final orbit, as determined by the above equations (see (52)) is $2K$, hence

$$v = 4\pi K a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (131)$$

and
$$v^2 = \beta^2 c^2 = 16\pi^2 K^2 a^2. \quad . \quad . \quad . \quad . \quad . \quad (132)$$

Eliminating v^2 between (129) and (132), we have

$$hK = 16\pi^2 m_0 K^2 a^2, \quad \text{or} \quad a^2 = \frac{h}{16\pi^2 m_0 K}. \quad (133)$$

Whence

$$a = \frac{1}{4\pi}\left(\frac{h}{m_0 K}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (134)$$

Substituting in this the known values of h, m_0 and K, namely

$$h = 6.547 \times 10^{-27}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (135)$$

$$m_0 = 0.90 \times 10^{-27}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (136)$$

and
$$K = 3.290 \times 10^{15}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (137)$$

we obtain numerically

$$a = .374 \times 10^{-10}\text{cm.}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (138)$$

and by (131),

$$v = 1.542 \times 10^8, \quad \text{and} \quad \beta = 0.00514. \quad . \quad (139)$$

It has been found by Dr. Bohr that the value of K, Rydberg's constant, may be expressed with a surprising degree of accuracy in terms of the properties of the electrons by the following formula:

$$K = \frac{2\pi^2 m_0 e^4}{h^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (140)$$

There are some coincidences as to the numerical value of K that are worth mentioning. In the first place the significant figures of K, 3.290, are very close indeed to the value of $\pi^2/3 = 3.289{,}868$, which plays such a prominent rôle in this theory. If we separate (140) into two factors, thus,

$$K = \frac{\pi^2}{3} \times \frac{6 m_0 e^4}{h^3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (141)$$

it is seen that the value of $6m_0e^4/h^3$ must be remarkably close to the even number 10^{15}. Using the values in (135) and (136), and putting

$$e = 4.774 \times 10^{-10}, \quad \ldots\ldots \quad (142)$$

we obtain

$$\frac{6m_0e^4}{h^3} = 0.99953 \times 10^{15}. \quad \ldots\ldots \quad (143)$$

So far as can be seen there is no theoretical support in this theory for the combination of quantities m_0, e and h in (140). It seems rather as if the Rydberg constant should be connected in some way with the properties of the nucleus of the atom. We have already seen that there is strong support for the Lorentz mass formula given in (3) above and repeated here.

$$m = \frac{4}{5a}\left(\frac{e}{c}\right)^2 \quad \text{or} \quad a = \frac{4}{5m}\left(\frac{e}{c}\right)^2. \quad \ldots \quad (144)$$

If this formula is applied to the nucleus of the hydrogen atom, of charge $2e$, and the following values are used,

$$m_H = 1.662 \times 10^{-24}, \quad \ldots\ldots \quad (145)$$

$$2e = 2 \times 4.774 \times 10^{-10}, \quad \ldots \quad (146)$$

$$c = 3 \times 10^{10}, \quad \ldots\ldots \quad (147)$$

we have

$$a_H = \frac{3.2}{1.662 \times 10^{-24}}\left(\frac{4.774 \times 10^{-10}}{3 \times 10^{10}}\right)^2$$

$$= 4.8756 \times 10^{-16}\text{cm}. \quad \ldots\ldots \quad (148)$$

The reciprocal of the expression $\frac{2}{m_H}\left(\frac{e}{c}\right)^2$, which occurs in this formula, is numerically nearly equal to K. If it is taken to be exactly equal numerically, we have a new relation

$$2K = m_H\left(\frac{c}{e}\right)^2 \quad \ldots\ldots \quad (149)$$

and

$$m_H = 2K\left(\frac{e}{c}\right)^2 \quad \ldots\ldots \quad (150)$$

Substituting in this the above value of e, and the value $K = 3.290 \times 10^{15}$, we obtain numerically

$$m_H = 1.666 \times 10^{-24}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (151)$$

If, however, the value of K of (149) is substituted in the expression for the radius (144), the m_H, e and c disappear, and we obtain the simple expression connecting the radius of the nucleus with K as follows:

$$a_H = \frac{8}{5K} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (152)$$

This gives numerically a slightly smaller radius than was obtained in (148) above, namely,

$$a_H = 4.8620 \times 10^{-16} \text{ cm.} \quad . \quad . \quad . \quad . \quad (153)$$

Let us assume that this is the correct value of the radius because it is derived from a theoretical relation involving but one constant, K, which is known with precision. This gives an exact value for e^2/m_H as is evident by solving both (144) and (149) for this quantity, giving

$$\frac{e^2}{m_H} = \frac{5}{16} a_H c^2 = \frac{c^2}{2K} = 1.36778 \times 10^5 \quad . \quad . \quad . \quad . \quad (154)$$

The decimal places are retained, although they have no particular meaning beyond the fourth significant figure, because we have to separate this result into the two factors, e and m_H.

There is an apparent difficulty here, however, in adopting the relation between a_H and K in (152) because it makes K have the dimensions of an inverse length apparently. The dimensions of K should be those of a frequency according to (22) above, that is to say, K should have the dimension T^{-1}.

It should be pointed out that the difficulty lies rather with the Lorentz formula (144), for the specific inductive

capacity of the medium, k, has been suppressed. According to the electrostatic system of units, we have the following dimensions:

$$e = M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}k^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad (156)$$

$$c = LT^{-1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (157)$$

$$e/c = M^{\frac{1}{2}}L^{\frac{1}{2}}k^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (158)$$

$$\frac{4}{5m}(e/c)^2 = Lk. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (159)$$

This is the quantity that appears on the right of (144), and on the left appears simply a, which has the dimension L, and not Lk. Or, in other words, the k is entirely ignored in this equation, and it is assumed that it is dimensionless. If we do not admit that this is so, and attribute to k some dimensions in terms of L and T, we may, by arbitrarily giving to k the dimensions of the reciprocal of a velocity, namely $L^{-1}T$, correct the apparent difficulty. The corrected equation is as follows:

$$a_H k = \frac{4}{5m_H}\left(\frac{2e}{c}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (160)$$

The dimensions of both sides of the equation now agree, and are equal to $LL^{-1}T = T$.

In a similar manner the k should be included with equation (152), making it

$$a_H k = 8/5K. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (161)$$

This makes the Rydberg constant, K, the reciprocal of a time, T^{-1}, as it should be. So long as k has a unit value, this change will make no difference in the numerical results already given.

It seems worth remarking before concluding the discussion of the new expression for K in (149) that, if we express the single charge, e, in electromagnetic units, in-

stead of electrostatic units, the velocity of light disappears, leaving the simple relation,

$$2K = \frac{m_H}{e^2} \text{ electromagnetic units.} \quad . \quad . \quad . \quad (162)$$

We have seen that the value of the Rydberg constant adopted in (149) gives an exact value of the ratio e^2/m_H, but it gives neither quantity individually. There is another experimental equation derived from experiments on the electrochemical equivalent of silver as follows:

$$\frac{e}{c}\frac{A_H}{m_H} = 9649.4, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (163)$$

the constant 9649.4 sometimes being called the Faraday constant. The A_H denotes the atomic weight of hydrogen referred to oxygen = 16. This equation gives an independent value of the ratio of e/m_H. By dividing the ratio e^2/m_H in (154) by the ratio e/m_H in (163) the m_H cancels, and the following value of e is obtained:

$$e = 4.763 \times 10^{-10}. \quad . \quad . \quad . \quad . \quad (164)$$

And using this value of e in (163) the value of m_H becomes,

$$m_H = 1.658 \times 10^{-24}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (165)$$

It is considered that these numerical values are both within the experimental error of determining them.

By the use of another experimental result, namely, the determination of the ratio of the charge to the mass of the electron, a value for the mass of the electron is determined. We have

$$\frac{e}{cm_0} = 1.767 \times 10^7. \quad . \quad . \quad . \quad . \quad . \quad (166)$$

The experimental constant 1.767×10^7 is that determined by Bucherer, and it may be referred to as

Bucherer's constant. In order to obtain a system of values that is consistent throughout with the theoretical formulæ adopted, we shall use the value of e as determined by these formulæ in (164), namely, 4.763×10^{-10}, giving

$$m_0 = .898 \times 10^{-27} \text{ grams.} \quad . \quad . \quad . \quad . \quad (167)$$

XI

LET us now return to the consideration of the velocity of the electrons in a ring of electrons as given by (129) above. If the value of K in (149) is substituted for the K in (129), we find

$$v^2 = \frac{b}{2}\frac{m_H}{m_0}\left(\frac{c}{e}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (168)$$

and

$$v = \frac{\sqrt{2}}{2}\frac{c}{e}\sqrt{\frac{bm_H}{m_0}} = \beta c, \quad . \quad . \quad . \quad . \quad . \quad (169)$$

whence

$$\beta = \frac{\sqrt{2}}{2e}\sqrt{\frac{bm_H}{m_0}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (170)$$

and

$$\beta^2 = \frac{bm_H}{2e^2m_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (171)$$

In this case the formula represents the velocity of an electron in a ring o two electrons, the 2 under the radical representing the number of electrons in the ring. If the 2 is replaced by p, as representing the number of electrons in any ring, the formula becomes

$$v = \frac{\sqrt{p}}{2}\frac{c}{e}\sqrt{\frac{bm_H}{m_0}} \quad \text{or} \quad \beta = \frac{\sqrt{p}}{2e}\sqrt{\frac{bm_H}{m_0}}. \quad . \quad . \quad (172)$$

The chief characteristic of this formula is that it makes the velocity of the electrons in the ring independent of the radius of the ring. It is considered that this is approximately true of any of the rings of electrons in atoms that have many rings. The speed of any ring is dependent only upon the number of electrons that it contains and is independent of the velocities that other rings

in the same atom may have. It is considered also that this formula is not exact but very approximate, and that the true velocity is dependent upon the radius but only to the second order. The reason for holding this view will be given in a subsequent section, where it is also pointed out how great the variation from this formula probably is.

This formula, and the dependency of the velocity of the electrons in a ring merely upon the number of electrons in the ring, is a radical departure from previous theories of the atom. It contains within it the idea that the cause of the revolution of the ring is the mutual action of the electrons in the ring upon each other. Electromagnetic theory shows that the force exerted upon a single electron in the ring by all of the other electrons always has a positive component along the tangent to the orbit in the direction of the motion. This tangential force must be reduced to zero by some equal and opposite force before there can be a steady and uniform motion of the ring. The force that is supposed to counterbalance this positive force along the tangent line is the tangential reaction of the electron being considered upon itself. The only way that this problem can be treated by electromagnetic theory is to make certain hypotheses concerning the electron itself, such, for example, as to assume that we have the solid Lorentz electron or some other form that has been proposed. The forces that are derived from these different forms of hypotheses differ according to the hypothesis, and the problem can have no certain and definite solution. The hypotheses that have to be made really beg the question.

It is not, therefore, sufficient to say that there is complete equilibrium and uniform velocity of the ring if we merely equate the forces normal to the orbit, or along

the radius, to zero. The particular velocity that this would result in may not be such as to cause the tangential forces of the other electrons in the ring to balance the force that the electron exerts upon itself at this velocity. If so, these tangential forces will alter the velocity until they do balance, and the radius will have to change accordingly until the radial forces also balance. The number of electrons in the ring may in this way control the speed.

There has always existed a difficulty in considering rings of electrons from the point of view of electromagnetic theory, for this theory shows that there is of necessity a certain amount of radiation of energy from a ring of electrons unless the number of electrons is very large. For example, if it is assumed that the rate of radiation of energy from a single electron in an orbit is unity according to the theory, then the rate of radiation from a ring of two electrons is about 4000 times smaller, and from a ring of three about forty million times smaller, and from a ring of four about a million million times smaller, and so on, the rate falling off with very great rapidity for a small increase in the number of electrons. In looking at this matter from the standpoint of the equations above given, and the theory as above outlined, it is seen that the final orbit really corresponds to the case where $\tau_2 = 0$ in (24). Putting $\tau_2 = 0$ in this we have

$$\nu = 2\pi K\left(\frac{1}{\tau^2} - \frac{1}{\tau^2}\right) = 0, \quad . \quad . \quad . \quad . \quad . \quad (173)$$

and there is no frequency of radiation. The same result is obtained if any one of the series of frequencies is examined, which, as we have seen, always end with a zero frequency when the final orbit is attained. It is certainly stretching the logic of the case to say that there

is no radiation of energy simply because the frequency of the radiation has been reduced to zero. By analogy, when the frequency of an alternating current is reduced to zero, the result is a steady current and not a zero current. And by further analogy, an alternating-current instrument may be conceived that will show no record of a direct current, although this is not a common form of these instruments. It may easily be imagined that the photographic plate and our eyes are such instruments, as regards the energy radiated, as will show nothing when the frequency is reduced too low. This would make the apparent energy radiated, as expressed by (108), reduce to zero when the frequency falls to zero in accordance with observations. The actual direct-current energy may still be present and escape all observation. It is exceedingly small anyway, if we may trust electromagnetic theory for its value. A revised form of electromagnetic theory, and there is little doubt that it will eventually be revised, seems likely to make this theoretical energy of radiation smaller than the present theory does. The reasons for holding this view will be given in a later section. There is no necessity, so far as can be seen, to make the energy radiated from a ring of electrons in the final steady orbit exactly zero except possibly the difficulty experienced in accounting for the source of the energy. The rate of radiation is probably so slow that the internal energy of the electron itself would be capable of sustaining it for lengths of time so great that it has been as yet impossible to detect any change. And again the change, when it comes, may be of the nature of a sudden change analogous to the sudden alterations in the atoms as they disintegrate, which would escape observation. This matter of the source of the energy has been seriously considered by writers on electromagnetic theory, but, so

far as any definite results are concerned they may be considered to be negligible. It is a pure speculation to imagine anything about it in the present state of our knowledge, but this account would have been deficient had all reference to these difficulties been omitted.

Let us next return to the formula for the radius of the orbit in (134). This may also be expressed in terms of the properties of the electrons without the quantity K. If the value of K of (167) is substituted in (134), we have

$$a = \frac{\sqrt{2}\,e}{4\pi}\,\frac{e}{c}\left(\frac{h}{m_0 m_H}\right)^{\frac{1}{2}} = .374 \times 10^{-8}\,cm. \quad . \quad . \quad . \quad (174)$$

This radius $.374 \times 10^{-8}$ for the ring of two in hydrogen is smaller than the smallest orbit for the single electron of the Bohr theory, which is about $.529 \times 10^{-8}$, the next orbit being 4 times greater.

The absolute value of the smallest orbit in hydrogen is of considerable interest and importance. A value so large as $.529 \times 10^{-8}$ has presented considerable difficulty, especially when there is but a single electron in an orbit, for it is possible to liquefy hydrogen, and in this state the average distance between the centers of the atoms may be calculated with considerable certainty from the density of the liquid. This distance is of the same order as the distances between atoms in crystals of various kinds, say between 2 and 3×10^{-8} centimeters. If the radius of the orbit of the electron can never be less than $.529 \times 10^{-8}$ cm., then its diameter is 1.058×10^{-8} cm., nearly half the distance between the centers of the atoms, assuming that this distance is 2×10^{-8} cm., or one third, assuming that the distance is 3×10^{-8} cm. When the electrons in adjacent atoms are at the nearest points of their orbits there would be considerable interference, due

to their mutual action upon each other, and this must be so great that the system cannot be regarded as stable.

This difficulty is greatly reduced by reducing the size of the radius about 30 %, but it is reduced very much more by having two instead of one electron in the orbit. The energy required to produce a disturbance in the orbit must for some cause, as yet not known, rise above a certain minimum value before radiation sets in, and, provided this limit is not reached, there will be no radiation. The hydrogen atom above described seems to be superior in this respect to the single electron atom.

XII

WE will next compute the orbits of the two electrons according to the equations, which have been given above, in order that a definite picture of the motion of the electrons in certain instances may be obtained. For this purpose let us first select the case of the "head" series, where $\tau_2 = \infty$, as being the simplest for computation, although there is considerable labor involved in obtaining any numerical curve because it is expressed as an infinite series of terms.

The energy radiated in this series is a maximum according to (108), being equal to $hkz = hK\frac{\pi^2}{6} = .2154 \times 10^{-10} \times 1.644{,}934 = 0.3543 \times 10^{-10}$ ergs.

The potential energy, V_0, required to separate the two electrons completely away from the nucleus is a minimum in this case, and it happens to be equal to hKz, the same value as the radiated energy.

In this series, where $\tau_2 = \infty$, we have

$$\mu = \nu = 2\pi K \frac{1}{\tau^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (175)$$

and $$\Sigma\mu = \Sigma\nu = 2\pi K \Sigma \frac{1}{\tau^2} = \frac{\pi^3}{3} K, \quad . \quad . \quad . \quad . \quad (176)$$

also $$\Sigma(\mu + \nu) = 2\Sigma\nu = \frac{2\pi^3}{3} K. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (177)$$

Let us first calculate the curve for the sum of the radii $\rho_2 + \rho_1$ as given in (27) above. The constant multiplier is $\frac{k_{\tau_2}}{2}$, which is given by (59). In this the $\Sigma\nu^2$, corresponding to $\tau_2 = \infty$, is given in the table (88). Hence, in this series the constant multiplier is equal to $a/1.082{,}323\pi K$ and the complete equation we have to compute is

$$\rho_2 + \rho_1 = \frac{a}{1.082{,}323\pi K}\Sigma\left\{\nu e^{-\nu t}(\cos \nu t)i - \nu e^{-\nu t}(\sin \nu t)j\right\}. \quad . \quad (178)$$

The initial value, when $t = 0$, is given by (37) above and again by (100). Using the values of x and z in the tables (88) and (77) corresponding to $\tau_2 = \infty$, we have the initial value

$$(\rho_2 + \rho_1)_0 = 2a\frac{z}{x}i = 2a\frac{1.644{,}943}{1.082{,}323}i = 3.039{,}636ai. \quad . \quad (179)$$

To obtain the first portion of the curve, near the time $t = 0$, we may develop $e^{-\nu t}\cos \nu t$ and $e^{-\nu t}\sin \nu t$ in series of powers of νt, giving the following values up to and including $(\nu t)^{10}$. This series development is required to facilitate the addition of the infinite number of sines and cosines expressed by the summation in (178), for this reduces the process to finding the sums of the powers of νt instead of the sines and cosines of these angles which differ for every value of τ.

$e^{-\nu t}\sin \nu t =$

$$
\begin{array}{ll}
\left[+1\right](\nu t) & = +1.000{,}000{,}000{,}0(\nu t) \\
+\left[-1\right](\nu t)^2 & = -1.000{,}000{,}000{,}0(\nu t)^2 \\
+\left[-\frac{1}{\underline{|3}}+\frac{1}{\underline{|2}\,\underline{|1}}\right](\nu t)^3 & = +0.333{,}333{,}333{,}3(\nu t)^3 \\
+\left[+\frac{1}{\underline{|1}\,\underline{|3}}-\frac{1}{\underline{|3}\,\underline{|1}}\right](\nu t)^4 & = 0.000{,}000{,}000{,}0(\nu t)^4 \\
+\left[+\frac{1}{\underline{|5}}-\frac{1}{\underline{|2}\,\underline{|3}}+\frac{1}{\underline{|4}\,\underline{|1}}\right](\nu t)^5 & = -0.033{,}333{,}333{,}3(\nu t)^5 \\
+\left[-\frac{1}{\underline{|1}\,\underline{|5}}+\frac{1}{\underline{|3}\,\underline{|3}}-\frac{1}{\underline{|5}\,\underline{|1}}\right](\nu t)^6 & = +0.011{,}111{,}111{,}1(\nu t)^6 \\
+\left[-\frac{1}{\underline{|7}}+\frac{1}{\underline{|2}\,\underline{|5}}-\frac{1}{\underline{|4}\,\underline{|3}}+\frac{1}{\underline{|6}\,\underline{|1}}\right](\nu t)^7 & = -0.001{,}587{,}301{,}6(\nu t)^7 \\
+\left[+\frac{1}{\underline{|1}\,\underline{|7}}-\frac{1}{\underline{|3}\,\underline{|5}}+\frac{1}{\underline{|5}\,\underline{|3}}-\frac{1}{\underline{|7}\,\underline{|1}}\right](\nu t)^8 & = 0.000{,}000{,}000{,}0(\nu t)^8 \\
+\left[+\frac{1}{\underline{|9}}-\frac{1}{\underline{|2}\,\underline{|7}}+\frac{1}{\underline{|4}\,\underline{|5}}-\frac{1}{\underline{|6}\,\underline{|3}}+\frac{1}{\underline{|8}\,\underline{|1}}\right](\nu t)^9 & = +0.000{,}044{,}091{,}6(\nu t)^9 \\
+\left[-\frac{1}{\underline{|1}\,\underline{|9}}+\frac{1}{\underline{|3}\,\underline{|7}}-\frac{1}{\underline{|5}\,\underline{|5}}+\frac{1}{\underline{|7}\,\underline{|3}}-\frac{1}{\underline{|9}\,\underline{|1}}\right](\nu t)^{10} & = -0.000{,}008{,}817{,}3(\nu t)^{10} \\
+\left[\text{etc.} \ldots\ldots\ldots\ldots\ldots\ldots\right. & \qquad\qquad (180)
\end{array}
$$

$e^{-\nu t}\cos \nu t =$

$$
\begin{array}{ll}
\left[+1\right](\nu t)^0 & = +1.000{,}000{,}000{,}0 \\
+\left[-1\right](\nu t)^1 & = -1.000{,}000{,}000{,}0(\nu t) \\
+\left[-\frac{1}{\underline{|2}}+\frac{1}{\underline{|2}}\right](\nu t)^2 & = 0.000{,}000{,}000{,}0(\nu t)^2 \\
+\left[+\frac{1}{\underline{|1}\,\underline{|2}}-\frac{1}{\underline{|3}}\right](\nu t)^3 & = +0.333{,}333{,}333{,}3(\nu t)^3 \\
+\left[+\frac{1}{\underline{|4}}-\frac{1}{\underline{|2}\,\underline{|2}}+\frac{1}{\underline{|4}}\right](\nu t)^4 & = -0.166{,}666{,}666{,}7(\nu t)^4 \\
+\left[-\frac{1}{\underline{|1}\,\underline{|4}}+\frac{1}{\underline{|3}\,\underline{|2}}-\frac{1}{\underline{|5}}\right](\nu t)^5 & = +0.033{,}333{,}333{,}3(\nu t)^5 \\
+\left[-\frac{1}{\underline{|6}}+\frac{1}{\underline{|2}\,\underline{|4}}-\frac{1}{\underline{|4}\,\underline{|2}}+\frac{1}{\underline{|6}}\right](\nu t)^6 & = 0.000{,}000{,}000{,}0(\nu t)^6 \\
+\left[+\frac{1}{\underline{|1}\,\underline{|6}}-\frac{1}{\underline{|3}\,\underline{|4}}+\frac{1}{\underline{|5}\,\underline{|2}}-\frac{1}{\underline{|7}}\right](\nu t)^7 & = -0.001{,}587{,}301{,}6(\nu t)^7 \\
+\left[+\frac{1}{\underline{|8}}-\frac{1}{\underline{|2}\,\underline{|6}}+\frac{1}{\underline{|4}\,\underline{|4}}-\frac{1}{\underline{|6}\,\underline{|2}}+\frac{1}{\underline{|8}}\right](\nu t)^8 & = +0.000{,}396{,}825{,}4(\nu t)^8 \\
+\left[-\frac{1}{\underline{|1}\,\underline{|8}}+\frac{1}{\underline{|3}\,\underline{|6}}-\frac{1}{\underline{|5}\,\underline{|4}}+\frac{1}{\underline{|7}\,\underline{|2}}-\frac{1}{\underline{|9}}\right](\nu t)^9 & = -0.000{,}044{,}091{,}6(\nu t)^9 \\
+\left[\text{etc.} \ldots\ldots\ldots\ldots\ldots\ldots\right. & 0.000{,}000{,}000{,}0(\nu t)^{10}
\end{array}
$$

$$(181)$$

We have to multiply each term in these series by ν, this being a factor in (178), and this raises the powers of ν in each term by unity. Since, by (175), in this series

$$\Sigma(\nu^n) = (2\pi K)^n \Sigma \frac{1}{\tau^{2n}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (182)$$

the sum of all the terms in this series to infinity may be found by the use of the table (67). Taking out the factor $2\pi K$, which appears in every term, we obtain

$$
\begin{aligned}
\rho_2 + \rho_1 = 2a/1.082{,}323 \{ \quad & 1.000{,}000{,}000 \; \Sigma\frac{1}{\tau^2} \; (2\pi Kt)^0 \\
& - 1.000{,}000{,}000 \; \Sigma\frac{1}{\tau^4} \; (\text{ “ })^1 \\
& 0.000{,}000{,}000 \; \Sigma\frac{1}{\tau^6} \; (\text{ “ })^2 \\
& 0.333{,}333{,}333 \; \Sigma\frac{1}{\tau^8} \; (\text{ “ })^3 \\
& - 0.166{,}666{,}667 \; \Sigma\frac{1}{\tau^{10}} \; (\text{ “ })^4 \\
& 0.033{,}333{,}333 \; \Sigma\frac{1}{\tau^{12}} \; (\text{ “ })^5 \\
& 0.000{,}000{,}000 \; \Sigma\frac{1}{\tau^{14}} \; (\text{ “ })^6 \\
& - 0.001{,}587{,}3 \; \Sigma\frac{1}{\tau^{16}} \; (\text{ “ })^7 \\
& 0.000{,}396{,}825{,}4 \Sigma\frac{1}{\tau^{18}} \; (\text{ “ })^8 \\
& - 0.000{,}044{,}091{,}6 \Sigma\frac{1}{\tau^{20}} \; (\text{ “ })^9 \\
& 0.000{,}000{,}000{,}0 \Sigma\frac{1}{\tau^{22}} \; (\text{ “ })^{10} \\
& \dots\dots\dots\dots\dots\dots \} i \\
+ 2a/1.082{,}323 \{ \quad & - 1.000{,}000{,}000 \; \Sigma\frac{1}{\tau^4} \; (2\pi Kt)^1 \\
& 1.000{,}000{,}000 \; \Sigma\frac{1}{\tau^6} \; (\text{ “ })^2 \\
& - 0.333{,}333{,}333 \; \Sigma\frac{1}{\tau^8} \; (\text{ “ })^3 \\
& 0.000{,}000{,}000 \; \Sigma\frac{1}{\tau^{10}} \; (\text{ “ })^4 \\
& 0.033{,}333{,}333 \; \Sigma\frac{1}{\tau^{12}} \; (\text{ “ })^5 \\
& - 0.011{,}111{,}111 \; \Sigma\frac{1}{\tau^{14}} \; (\text{ “ })^6 \\
& 0.001{,}586{,}3 \; \Sigma\frac{1}{\tau^{16}} \; (\text{ “ })^7 \\
& 0.000{,}000{,}000 \; \Sigma\frac{1}{\tau^{18}} \; (\text{ “ })^8 \\
& - 0.000{,}044{,}091{,}6 \Sigma\frac{1}{\tau^{20}} \; (\text{ “ })^9 \\
& 0.000{,}009{,}817{,}3 \Sigma\frac{1}{\tau^{22}} \; (\text{ “ })^{10} \\
& \dots\dots\dots\dots\dots\dots \} j \quad . \quad (183)
\end{aligned}
$$

Multiplying in the values of $\Sigma \frac{1}{\tau^n}$ according to the table (67) gives the following:

$$
\begin{aligned}
\rho_2 + \rho_1 = 2a/1.082,323\{\; & 1.644,934 && (2\pi Kt)^0 \\
& -1.082,323 && (\;``\;)^1 \\
& 0.000,000 && (\;``\;)^2 \\
& 0.334,692 && (\;``\;)^3 \\
& -0.166,832,5 && (\;``\;)^4 \\
& 0.033,415,3 && (\;``\;)^5 \\
& 0.000,000,0 && (\;``\;)^6 \\
& -0.001,587,32 && (\;``\;)^7 \\
& 0.000,396,826,9 && (\;``\;)^8 \\
& -0.000,044,916,4 && (\;``\;)^9 \\
& 0.000,000,000,0 && (\;``\;)^{10} \\
& \dots\dots\dots\dots && \}i \\
+2a/1.082,323\{\; & -1.082,323 && (2\pi Kt)^1 \\
& 1.017,343 && (\;``\;)^2 \\
& -0.334,692 && (\;``\;)^3 \\
& 0.000,000 && (\;``\;)^4 \\
& 0.033,415,3 && (\;``\;)^5 \\
& -0.011,111,8 && (\;``\;)^6 \\
& 0.001,587,32 && (\;``\;)^7 \\
& 0.000,000,00 && (\;``\;)^8 \\
& -0.000,044,916,4 && (\;``\;)^9 \\
& 0.000,008,817,3 && (\;``\;)^{10} \\
& \dots\dots\dots\dots && \}j \;\dots\; (184)
\end{aligned}
$$

From this it is seen that, when $t = 0$, $\rho_2 + \rho_1 = 3.039,636ai$, but that the series is not useful beyond a certain value of t. The highest term of the series has a frequency K, and a period $1/K$. If $t = 1/K$, or $Kt = 1$, this one component has executed one revolution. If we set $t = 1/2\pi K$, or $Kt = 1/2\pi$, the highest frequency term has passed through $1/2\pi$th of a revolution, or about 57°. The curve may be calculated up to this point of time without sensible error due to the convergency of the series. Letting $2\pi Kt$ take in succession the values of .1, .2, .3, etc., up to 1.0, the following numerical values are obtained.

	$(\rho_2+\rho_1)\Big/\dfrac{2a}{1.082{,}323}$		$(\rho_2+\rho_1)/a$	
$2\pi Kt$	i	j	i	j
.0	1.644,934	0.000,000	3.039,636	0.000,000
.1	1.537,020	− 0.068,294	2.840,224	− 0.126,198
.2	1.431,004	− 0.178,342	2.644,319	− 0.329,554
.3	1.328,003	− 0.242,099	2.453,986	− 0.447,369
.4	1.299,494	− 0.291,275	2.271,953	− 0.538,240
.5	1.136,215	− 0.327,779	2.099,585	− 0.605,695
.6	1.048,718	− 0.353,320	1.937,902	− 0.652,892
.7	0.967,558	− 0.369,483	1.787,928	− 0.682,759
.8	0.892,781	− 0.377,757	1.649,747	− 0.698,048
.9	0.824,501	− 0.379,471	1.523,576	− 0.701,216
1.0	0.762,650	− 0.375,817	1.409,284	− 0.694,464 . (185)

Charting the equation (184) in this way not only gives a curved path but locates points upon the curve at equal intervals of time, from which the velocity of the point in the curve is at once apparent. The curve is shown as curve I, Fig. 5.

Let us next calculate the value of ρ_1, from which, and the value of $\rho_2 + \rho_1$ already found, we may find ρ_2 and $\rho_2 - \rho_1$ graphically without the necessity for arithmetical calculation. Using the lower signs in equation (34) we obtain $2\rho_1$; but it may be remarked that in the special case we are now considering, that of the head series, where $\nu = \mu$, the first two or k_{τ_2}-terms exactly cancel, leaving only the B_{τ_2} and A-terms. The value of B_{τ_2} is now, by (61)

$$B_{\tau_2} = \frac{8\pi Ka}{s_2}\left(1 + \frac{y}{x}\right), \quad \dots \dots \quad (186)$$

where now $y = x$, and $s_2 = (4\pi K)^2 \times 1.082{,}323$. Whence

$$B_{\tau_2} = a/1.082{,}323\pi K, \quad \dots \dots \quad (187)$$

which is the same as $\frac{k_{\tau_2}}{2}$, the coefficient in (184) above.

Hence

$$\rho_1 = \frac{a}{2 \times 1.082,323\pi K} \Sigma \left\{ \nu e^{-2\nu t}[(\cos 2\nu t)i - (\sin 2\nu t)j] \right\}$$

$$- a[(\cos 2\nu t)i - (\sin 2\nu t)j]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (188)$$

The values of $e^{-\nu t} \sin \nu t$ and $e^{-\nu t} \cos \nu t$ given in (180) and (181) above will also answer for this, if 2ν is substi-

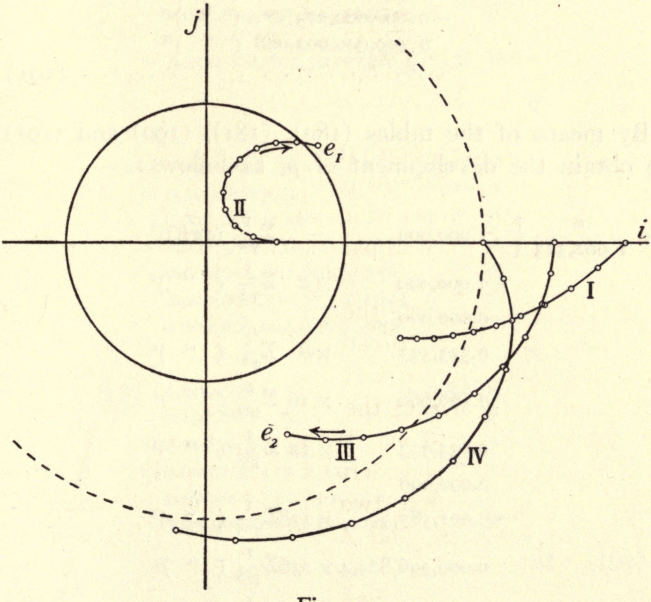

Fig. 5.

tuted in the place of ν. Besides these are required for the last term the expansion of $\sin 2\nu t$ and $\cos 2\nu t$, which follow.

$$\begin{aligned}
\sin(\mu+\nu)t = \sin 2\nu t = \ & 1.000,000,0 && (4\pi Kt)^1 \\
& -0.166,666,6 && (\ \text{''}\ )^3 \\
& 0.008,333,3 && (\ \text{''}\ )^5 \\
& -0.000,198,4 && (\ \text{''}\ )^7 \\
& 0.000,002,756 && (\ \text{''}\ )^9 \\
& -0.000,000,025,052 && (\ \text{''}\ )^{11} \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots && (190)
\end{aligned}$$

$$\begin{aligned}
\cos(\mu+\nu)t = \cos 2\nu t = \ & 1.000,000,0 && (4\pi Kt)^0 \\
& -0.500,000,0 && (\ \text{''}\ )^2 \\
& 0.041,666,6 && (\ \text{''}\ )^4 \\
& -0.001,388,8 && (\ \text{''}\ )^6 \\
& 0.000,024,802 && (\ \text{''}\ )^8 \\
& -0.000,000,275,57 && (\ \text{''}\ )^{10} \\
& 0.000,000,002,088 && (\ \text{''}\ )^{12} \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots && (191)
\end{aligned}$$

By means of the tables (180), (181), (190) and (191) we obtain the development of ρ_1 as follows:

$$\begin{aligned}
\rho_1 = \frac{a}{1.082,323}\Bigg\{ \ & 1.000,000 && && \Sigma\frac{1}{\tau^2}\ (2\pi Kt)^0 \\
& -1.000,000 && \times 2 && \Sigma\frac{1}{\tau^4}\ (\ \text{''}\ )^1 \\
& 0.000,000 && && \\
& 0.333,333 && \times 8 && \Sigma\frac{1}{\tau^8}\ (\ \text{''}\ )^3 \\
& -0.166,666 && \times 16 && \Sigma\frac{1}{\tau^{10}}\ (\ \text{''}\ )^4 \\
& 0.033,333 && \times 32 && \Sigma\frac{1}{\tau^{12}}\ (\ \text{''}\ )^5 \\
& 0.000,000 && && \\
& -0.001,587,3 && \times 128 && \Sigma\frac{1}{\tau^{16}}\ (\ \text{''}\ )^7 \\
& 0.000,396,825,4 && \times 256 && \Sigma\frac{1}{\tau^{18}}\ (\ \text{''}\ )^8 \\
& -0.000,044,091,6 && \times 512 && \Sigma\frac{1}{\tau^{20}}\ (\ \text{''}\ )^9 \\
& 0.000,000,000,0 && && \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \Big\}\, i
\end{aligned}$$

$$
\begin{array}{rlll}
-\dfrac{a}{1.082,323}\Bigg\{ & 1.000,000 & \times 2\ \Sigma\dfrac{1}{\tau^4} & (2\pi Kt)^1 \\
& -\ 1.000,000 & \times 4\ \Sigma\dfrac{1}{\tau^6} & (\ \text{"}\ )^2 \\
& 0.333,333 & \times 8\ \Sigma\dfrac{1}{\tau^8} & (\ \text{"}\ )^3 \\
& 0.000,000 & & \\
& -\ 0.033,333 & \times 32\ \Sigma\dfrac{1}{\tau^{12}} & (\ \text{"}\ )^5 \\
& 0.011,111 & \times 64\ \Sigma\dfrac{1}{\tau^{14}} & (\ \text{"}\ )^6 \\
& -\ 0.001,587,3 & \times 128\Sigma\dfrac{1}{\tau^{16}} & (\ \text{"}\ )^7 \\
& 0.000,000,0 & & \\
& 0.000,044,091,6 & \times 512\Sigma\dfrac{1}{\tau^{20}} & (\ \text{"}\ )^9 \\
& -\ 0.000,008,817,3 & \times 1024\Sigma\dfrac{1}{\tau^{22}} & (\ \text{"}\ )^{10} \\
& \ldots\ldots\ldots\ldots & & \Big\}j
\end{array}
$$

$$
\begin{array}{rll}
-a\Big\{ & 1.000,000 & (2\pi Kt)^0 \\
& -\ 0.500,000 \times 4 & (\ \text{"}\ )^2 \\
& 0.041,666 \times 16 & (\ \text{"}\ )^4 \\
& -\ 0.001,388 \times 64 & (\ \text{"}\ )^6 \\
& 0.000,024,801,625 \times 256 & (\ \text{"}\ )^8 \\
& -\ 0.000,000,275,573 \times 1024 & (\ \text{"}\ )^{10} \\
& 0.000,000,002,088 \times 4096 & (\ \text{"}\ )^{12} \\
& \ldots\ldots\ldots\ldots & \Big\}i \\
+a\Big\{ & 1.000,000 \times 2 & (2\pi Kt)^1 \\
& -\ 0.166,666 \times 8 & (\ \text{"}\ )^3 \\
& 0.008,333 \times 32 & (\ \text{"}\ )^5 \\
& -\ 0.000,198,413 \times 128 & (\ \text{"}\ )^7 \\
& 0.000,002,755,73 \times 512 & (\ \text{"}\ )^9 \\
& -\ 0.000,000,025,052 \times 2048 & (\ \text{"}\ )^{11} \\
& \ldots\ldots\ldots\ldots & \Big\}j\ .
\end{array} \qquad (192)
$$

Multiplying in the numerical values of $\Sigma\frac{1}{\tau^2}$, $\Sigma\frac{1}{\tau^4}$, etc., according to the table (67), and the constant $1/1.082,323$, we finally obtain

$$
\begin{array}{rlll}
\rho_1 = a\{ & 1.519,817,75 & (2\pi Kt)^0 & \\
 & -\,2.000,000,00 & (\text{ `` })^1 & \\
 & 0.000,000,00 & (\text{ `` })^2 & \\
 & 2.473,881,668,7 & (\text{ `` })^3 & \\
 & -\,2.466,286,196,6 & (\text{ `` })^4 & \\
 & 0.985,776,820,1 & (\text{ `` })^5 & \\
 & 0.000,000,000,0 & (\text{ `` })^6 & \\
 & -\,0.187,723,499 & (\text{ `` })^7 & \\
 & 0.093,860,767,9 & (\text{ `` })^8 & \\
 & -\,0.020,857,836,2 & (\text{ `` })^9 & \\
 & 0.000,000,000,0 & (\text{ `` })^{10} & \\
 & \ldots\ldots\ldots\ldots & & \}i \\
+\,a\{ & -\,2.000,000,000,0 & (2\pi Kt)^1 & \\
 & 3.759,848,283,8 & (\text{ `` })^2 & \\
 & -\,2.473,881,668,7 & (\text{ `` })^3 & \\
 & 0.000,000,000,0 & (\text{ `` })^4 & \\
 & 0.985,776,820,1 & (\text{ `` })^5 & \\
 & -\,0.657,042,959,4 & (\text{ `` })^6 & \\
 & 0.187,723,499 & (\text{ `` })^7 & \\
 & 0.000,000,000 & (\text{ `` })^8 & \\
 & -\,0.020,857,836,2 & (\text{ `` })^9 & \\
 & 0.008,342,163,5 & (\text{ `` })^{10} & \\
 & \ldots\ldots\ldots\ldots & & \}j \\
+\,a\{ & -\,1.000,000,000 & (2\pi Kt)^0 & \\
 & 2.000,000,000 & (\text{ `` })^2 & \\
 & -\,0.666,666,666 & (\text{ `` })^4 & \\
 & 0.088,888,888 & (\text{ `` })^6 & \\
 & -\,0.006,349,216 & (\text{ `` })^8 & \\
 & 0.000,282,187 & (\text{ `` })^{10} & \\
 & \ldots\ldots\ldots\ldots & & \}i \\
+\,a\{ & 2.000,000,000 & (2\pi Kt)^1 & \\
 & -\,1.333,333,333 & (\text{ `` })^3 & \\
 & 0.266,666,666 & (\text{ `` })^5 & \\
 & -\,0.025,396,86 & (\text{ `` })^7 & \\
 & 0.001,410,93 & (\text{ `` })^9 & \\
 & \ldots\ldots\ldots\ldots & & \}j \quad \ldots\ldots \quad (193)
\end{array}
$$

Adding the two sets of i and j terms in the above, we obtain

$$
\begin{array}{rlll}
\rho_1 = a\{ & 0.519{,}817{,}75 & (2\pi Kt)^0 & \\
 & 2.000{,}000{,}00 & (\;``\;)^1 & \\
 & 2.000{,}000{,}00 & (\;``\;)^2 & \\
 & 2.473{,}881{,}668{,}7 & (\;``\;)^3 & \\
 & 3.132{,}952{,}863{,}3 & (\;``\;)^4 & \\
 & 0.985{,}776{,}820{,}1 & (\;``\;)^5 & \\
 & 0.088{,}888{,}888{,}8 & (\;``\;)^9 & \\
 & 0.187{,}723{,}499 & (\;``\;)^7 & \\
 & 0.087{,}511{,}551{,}9 & (\;``\;)^8 & \\
 & -\,0.020{,}857{,}836{,}2 & (\;``\;)^9 & \\
 & 0.000{,}282{,}187 & (\;``\;)^{10} & \\
 & \ldots\ldots\ldots\ldots & & \}i \\
+\,a\{ & 0.000{,}000{,}000{,}0 & (2\pi Kt)^1 & \\
 & 3.759{,}849{,}283{,}8 & (\;``\;)^2 & \\
 & -\,3.807{,}215{,}002{,}0 & (\;``\;)^3 & \\
 & 0.000{,}000{,}000{,}0 & (\;``\;)^4 & \\
 & 1.252{,}443{,}486{,}8 & (\;``\;)^5 & \\
 & -\,0.657{,}042{,}959{,}4 & (\;``\;)^6 & \\
 & 0.162{,}326{,}639 & (\;``\;)^7 & \\
 & 0.000{,}000{,}000 & (\;``\;)^8 & \\
 & -\,0.019{,}446{,}906{,}2 & (\;``\;)^9 & \\
 & 0.008{,}342{,}163{,}5 & (\;``\;)^{10} & \\
 & \ldots\ldots\ldots\ldots & & \}j \;\ldots\ldots \quad (194)
\end{array}
$$

From this the following table of numerical values of (ρ_1 /a) have been obtained for a series of values of $2\pi Kt$, namely, .0, .1, .2, .3, etc., up to 1.0.

$2\pi Kt$	ρ_1/a	
	i	j
.0	0.519,818	0.000,000
.1	0.341,988	0.033,803
.2	0.214,627	0.120,297
.3	0.143,660	0.238,192
.4	0.128,122	0.368,310
.5	0.164,411	0.495,533
.6	0 244,953	0.602,321
.7	0.361,026	0.682,852
.8	0.502,053	0.727,498
.9	0.656,972	0 733,409
1.0	0.814,625	0 699,257

(195)

The curve obtained from this table is charted as curve II in Fig. 5. It represents the path followed by the first electron from the moment when the radiation of energy begins up to a time $t = 1/2\pi K$. Since we already have the sum of ρ_2 and ρ_1 in curve I, the path of the second electron is obtained by subtracting ρ_1 from $\rho_2 + \rho_1$. The position vector of e_2 is then equal to the line joining a given point in curve II with the corresponding point in curve I at the same time. These vectors, when transferred to the origin, or the nucleus of the atom, give the path of e_2 as curve III. The difference between ρ_2 and ρ_1 is the vector from a given point in curve II to the corresponding point of curve III, and this gives the curve IV, Fig. 5.

It may be seen from these curves that the paths of both e_1 and e_2 are approaching the small circular orbit, the full line, and will eventually arrive at the opposite ends of a common diameter of this orbit. The curve I, representing the sum of the position vectors, approaches the origin, finally becoming zero when radii of e_1 and e_2 are equal and opposite. The curve of the difference, IV, however, approaches a circular orbit of double the diameter of the final orbit of the electrons, which is shown by the larger circle.

It is to be regretted that these curves have not been computed for a greater distance than they have been. This may, of course, be done, but the series which have been developed will have to be abandoned on account of their non-convergence. By omitting the first periodic term from the series, which has the greatest frequency, and by computing it separately, and then computing the rest of the series as we have done above, the result may be considerably extended in time. The computation of the first term may be added to the rest after they are sep-

arately obtained. The labor involved in these computations is, however, considerable, as will be evident from an inspection of the work involved in the one example that has been given.

It will be interesting to observe the differences between the curves obtained from the different spectral series. The example that has been computed is the case of the head series only. It is necessary to content ourselves with this one example at present. The next case that would naturally be computed is that where $\tau_2 = 1$.

XIII

IT is proposed in the following sections to give some account of the results which have been obtained by considering atoms in their first state when neither radiating nor absorbing energy. It will be recognized at once that, if we knew the correct expression for the mechanical force with which one moving electrical charge acts upon another for any kind of motion, it should be possible to assume that the motion is circular motion such as we suppose the electrons have in the normal undisturbed state of all atoms. By applying these results to the electrons in the atoms it is conceivable that we may by adding up the effects of the individual electrons in an atom eventually arrive at the nature of the forces that atoms exert upon each other. The fundamental problem is, therefore, to obtain an expression for the mechanical force that a single electron revolving in a circle exerts upon another revolving in a different circle of different radius and different frequency.

The author has solved this problem by the use of two of the forms of electromagnetic theory that have been proposed, first[1] by the equations given by J. J. Thomson in his paper of 1881, and second[2] by the equations due

[1] A. C. Crehore, *Phil. Mag.*, Vol. XXVI, July, 1913, p. 58. Also *Phil. Mag.*, Vol. XXIX, June, 1915, p. 750; *Phil. Mag.*, Vol. XXX, August, 1915, p. 257.

[2] A. C. Crehore, *Phys. Rev.*, N. S., Vol. IX, No. 6, June, 1917, p. 445.

to Lorentz, which represent the current form of the equations of this theory. Electromagnetic theory has passed through several important stages of development since the early days when Maxwell published his celebrated treatise. And the process of this development is not at an end by any means as yet. It should not end until the results obtained from the theory are in complete harmony with all the facts of observation. A recent valuable contribution[1] has been made to this theory by Mega Nad Saha, who makes use of the modern four-dimensional analysis of Minkowski. This investigator arrives at equations having greater generality than those of Lorentz, which seem likely to have an important bearing upon the problem before us. We shall outline the results obtained by the author by the use of the Lorentz form of equations only, omitting any reference to the Thomson equations. The possible modifications that will be permitted by the use of the Saha equations have not yet been investigated. Reference must be had to the original publications for a detailed account of the derivation of the equations which we shall use here merely as the result there obtained. The complete equation[2] expressing the force that a second electron revolving in a circular orbit exerts upon a first electron in another orbit is too long to repeat here. The force is a variable force with time as the two electrons revolve about their orbits. If, however, these two electrons are in fixed orbits in atoms, the effect which they individually contribute to the attraction or the repulsion of the atoms must depend upon the average value of the variable force, averaged for time. The average force obtained from this equation, when resolved along the center line

[1] Mega Nad Saha, *loc. cit.*

[2] *Loc. cit.*, equations (48), (49), and (50), pp. 453, 454.

of the two orbits, is a very simple expression, namely:[1]

$$F_r = \tfrac{1}{2}e^2\beta_2^2[1 - (-X \sin \alpha + Z \cos \alpha)^2]r^{-2} \quad . \quad (196)$$

This denotes the force that the second electron exerts upon the first. The β_2 is the velocity of the second electron and e is the charge. r is the distance between the centers of the orbits supposed to be fixed and constant. The angle α is the angle between the directions of their axes of revolution, and the X and Z are the direction cosines defining the position of the center of the orbit of the second electron with reference to a set of rectangular axes, i, j, and k, through the center of the orbit of the first electron.

It is to be remarked first that the velocity of the first electron does not appear in this equation at all. That is to say, the force upon the first electron is, according to this result, entirely independent of its own velocity, and it does not matter what it is doing. As a direct result of this it may be shown that the force upon the first electron due to the second one may be entirely different from the force exerted upon the second due to the first. We would obtain the force on the second electron due to the first by putting β_1 in place of β_2 in the equation, and, if they were not equal to each other, the action and reaction would be unequal. It should be stated here that this equation gives the forces due to the velocity of motion of the electrons only, and becomes zero when the velocity is zero. There are besides these forces the large electrostatic forces[2] which have purposely been omitted from the equation because the electrostatic part cancels out when the positive nucleus of the atoms is also taken into the account.

The equation represents only the first term of an in-

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[1] *Loc. cit.*, equation (54), p. 456. [2] *Loc. cit.* See top of p. 456.

finite series of terms, r^{-3}, r^{-4}, etc., and at great distances the third and higher powers of r become so small that all terms except the first are negligible, that is to say, the equation as it stands is supposed to apply to the two electrons only when they are at a great distance apart as compared with the diameters of their orbits. There is one more qualification that has to be made as to this equation. The average given in (196) was obtained under the supposition[1] that the Doppler factor,

$$A = \frac{\partial t}{\partial \tau} = 1 - \frac{q_2 \cdot R}{cR}, \quad . \quad . \quad . \quad . \quad (197)$$

is a constant so nearly equal to unity that it cannot affect the average, for $1/A^3$ occurs as a factor of the original equation. On this point the author has been taken to task in a long article by G. A. Schott,[2] who has shown that the supposition that the Doppler factor is unity gives a different average force[3] from the supposition that it is variable, as given in the equation last above. On the hypothesis, which the author made, that this factor is sensibly equal to unity, Schott has also verified the result given above in (196). The result obtained by Schott on the supposition of A variable also gives a force that varies as the inverse square of the distance, but the magnitude differs in sign and depends upon β^4 instead of β^2, as in the equation given above.

The chief result, which has now been established by means of these deductions from the current form of electromagnetic theory by these investigations, is that

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[1] *Loc. cit.* See top of p. 455.

[2] G. A. Schott, *Phys. Rev.*, Sec. Ser., Vol. XII, July, 1918, p. 23. See also the author's reply to Schott, *Phys. Rev.*, Vol. XIII, No. 2, February, 1919, p. 89.

[3] *Loc. cit.*, the Schott paper, p. 37, equation (50). Also p. 91, equation (1), author's reply.

this theory demands that there be a force, which one revolving electron exerts upon another at a great distance, omitting all electrostatic forces because they eventually cancel, that shall vary as the inverse square of the distance law. If this result is applied to a great multitude of electrons such as make up the sum total of all the electrons in all of the atoms of a material mass of matter, the process of summing up these forces does not change in any way the character of the law of variation of the force with the distance. It may, therefore, be said that the above investigation demands that there be a force between two material massive bodies at a great distance from each other that varies inversely as the square of the distance. Now, the chief force that we know actually exists between any two bodies at a distance is the gravitational force which obeys the inverse square of the distance law. If, therefore, these deductions from the theory are not in harmony in all respects with the gravitational force, we are forced to conclude that something is amiss with the theory. On this account the author has made a careful comparison between the result of the theory and the actual gravitational force which is known with precision. It is very significant indeed that the equation (196) above is in complete agreement with the gravitational law[1] in every respect but one, and this is the magnitude of the force. There are several other checks besides the magnitude of the force that it must fulfil. They are as follows. It must make the force proportional to the product of the masses; it must show that the force is always an attraction and never a repulsion; it must show that the force is independent of the orientation of the two bodies, whether they be crystals

[1] A. C. Crehore, *Phys. Rev.*, N. S., Vol. XII, No. 1, July, 1918, p. 13.

or any other of the forms of matter, solids, liquids, or gases.

It will presently be shown that this equation agrees in all these respects in a remarkable manner with the law of gravitation, except only in the magnitude; and the Schott equation, obtained by assuming the Doppler factor variable, does not agree in any respect save that it requires the inverse square of the distance law.

As to the magnitude of the force the equation is evidently deficient because it does not agree with one of the most fundamental laws of Nature, one of Newton's laws, that of equal Action and Reaction. The equation as it stands makes it possible that the attraction of the body A for the body B may differ from the attraction of the body B for the body A. This is because the two velocities β_1 and β_2 do not occur symmetrically in the equation. If it is found that we can make use of this equation, and, by the simple expedient of correcting it by the use of a constant multiplier, make it agree with gravitational law in all respects, then there are strong grounds for believing that this constant multiplier should have existed in the correct form of electromagnetic equations from which this was derived. And in this manner it is hoped that this experimental check of the electromagnetic theory by comparing its results with known facts may be of material assistance in eventually revising the theory.

We shall, therefore, arbitrarily introduce a multiplying factor to correct the magnitude of the force expressed in (196) and shall then proceed to determine the required numerical value of this factor to make it agree with the gravitational law. However, it is conjectured that one of the factors must be β_1^2 in order to make the equation conform to the law of equal Action and Reaction. Let us temporarily denote the rest of the multiplying factor

by x, signifying an unknown quantity to be determined, and make the whole factor $\beta_1^2 x$. To anticipate the result of the determination of x, it may now be stated that numerically x comes out equal to $.8625 \times 10^{-27}$. This value is so close to the value of the mass of the electron itself given in (167) above, namely $.898 \times 10^{-27}$ that we have strong grounds for thinking that the true multiplying factor should be $m_0\beta_1^2$.

If there should be any multiplying factor at all required for this equation, we must expect that it will have some value connecting it in a very simple manner with the properties of the electron, and this factor satisfies this demand in a very complete manner. But, if this is the true factor, it immediately raises the question, how is it possible to multiply the expression on the right of equation (196) by the quantity $m_0\beta_1^2$ and still have the expression represent a force. For the quantities on the right of the equation should already have the dimensions of a force, and the result of the multiplication is to make the dimensions a force times a mass. This is a most important consideration, indeed, and it leads again to some most important results, as will be shown.

It was shown above, in considering the Lorentz mass formula (144), that it is customary by writers on the modern electromagnetic theory to suppress the quantity k, the specific inductive capacity of the medium, regarding it as equal to unity and as being dimensionless. We corrected this equation by introducing the k, writing it as in (169) in order to make the dimensions of the two members of this equation the same, on the supposition that k is not dimensionless, but that it has dimensions in terms of length and time. And this necessitated that ak on the left of the equation (161) should have the dimensions of the reciprocal of the Rydberg constant, namely

that of time alone. This result, making the radius times the specific inductive capacity have the dimensions of time, required that the dimensions of k should be those of the reciprocal of a velocity. This is a rational result because we already know that the product of k and μ, the magnetic permeability, are those of the reciprocal of the square of a velocity, namely the velocity of light, for the equation

$$k\mu = \frac{1}{c^2} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (198)$$

has been known for some time; but neither the dimensions of k nor of μ separately have been known. The fact that we possess this equation for the product of k and μ in itself shows that both k and μ should have some dimensions in terms of length and of time. And since we have fixed upon the dimensions of k as being those of the reciprocal of a velocity, the dimensions of μ in terms of length and of time are automatically determined by the equation (198). This gives μ the same dimensions as k, namely the reciprocal of a velocity. It also makes the ratio of k to μ dimensionless, since it comes out the ratio of two velocities, thus having the same kind of dimensions as β, the ratio of the velocity of an electron to that of light.

This matter has been referred to again here because an examination of the original electromagnetic equation, from which (196) has been derived, shows that the k has again been suppressed. This becomes apparent when we write out the dimensions of the quantities on the right of the equation, which are the same evidently as the dimensions of e^2/r^2.

The dimensions of e on the electrostatic system of units are given by (156) above, and the dimensions of e^2/r^2 are. therefore,

$$e^2/r^2 = LMT^{-2}k. \quad . \; . \; . \; . \; . \; . \quad (199)$$

The dimensions of force are

$$F = LMT^{-2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (200)$$

The two members of the equation as it now stands do not, therefore, have the same dimensions. Some multiplying factor having some physical dimensions is in fact required in order to correct the dimensions of the equation, assuming that k is not dimensionless in terms of L and T. This factor we shall assume is the $x\beta_1^2$, or its equivalent, as we shall prove, $m_0\beta_1^2$, having the the dimensions of mass. The complete revised equation now becomes

$$F_r = \tfrac{1}{2}e^2x\beta_1^2\beta_2^2[1 - (-X\sin\alpha + Z\cos\alpha)^2]r^{-2}, \quad (201)$$

where the x may be taken as equivalent to the mass of the electron, m_0.

The dimensions of the quantities on the right of the equation are the same as the dimensions of e^2m_0/r^2, and these dimensions according to (199) are

$$LMT^{-2}Mk. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (202)$$

But these dimensions must be those of force, LMT^{-2}. Hence Mk must have zero dimensions[1] in terms of L and T. But we have already made the dimensions of k those of the reciprocal of a velocity. Hence, to satisfy this equation, the dimensions of mass must be the reciprocal of k and equal to those of velocity, namely LT^{-1}.

We have by these means thus come to the conclusion that mass, specific inductive capacity, and magnetic permeability are not fundamental units like length and time, but that they may be expressed in terms of length and time. Very strong confirmation of these ideas is to

[1] *Loc. cit., Phys. Rev.*, June, 1917, p. 464, equation (77).

be found in the results obtained by eliminating M, k and μ from the common tables of dimensions of quantities as ordinarily given in an electrostatic system and in an electromagnetic system. Such a table is given on the following page for some of the more common units, and a reduction is made to a new system which may be called the L-T, or the space-time, system, in which all dimensions of every kind of quantity are expressed merely in terms of space and time.

The reduction of a unit in either system, electrostatic or electromagnetic, to the space-time system is effected by substituting the values above determined, namely,

$$k = \mu = L^{-1}T \quad \ldots\ldots\ldots \quad (203)$$

and

$$M = LT^{-1} \quad \ldots\ldots\ldots\ldots \quad (204)$$

It is to be noticed, first, that each unit, whether reduced from the electrostatic system or from the electromagnetic system comes out of the same dimensions in terms of L and T. This is as it should be if we are to regard dimensions as characteristic of a quantity. Indeed, the common systems are misleading in regard to this and make it appear that there is nothing definite about the dimensions so far as length and time are concerned, although this is in appearance only.

In the second place it will be noticed that several of the quantities which have formerly been regarded as different things have the same dimensions in the space-time system. And these quantities so reduced to the same dimensions do not come out in a haphazard fashion, but they are the very quantities that we have already suspected were of the same nature and therefore might be expected to have the same dimensions. For example, quantity of electricity, or electrical charge, and quantity of magnetism, or the strength of a magnetic pole, receive

Kind of Quantity	Symbol	Dimensions of electrostatic system of units. Exponents only expressed.				Dimensions of electromagnetic system of units. Exponents only expressed.				Ratio electrostatic to electromagnetic units. Exponents only expressed.					Dimensions of length and time system of units. Exponents expressed.	
		L	*M*	*T*	*k*	*L*	*M*	*T*	μ	*L*	*M*	*T*	*k*	μ	*L*	*T*
Mass......	*m*	0	1	0	0	0	1	0	0	0	0	0	0	0	1	−1
Specific inductive capacity.	*K*	0	0	0	1	−2	0	2	−1	2	0	−2	1	1	−1	1
Permeability ...	μ	−2	0	2	−1	0	0	0	1	−2	0	2	−1	−1	−1	1
Momentum	*mv*	1	1	−1	0	1	1	−1	0	0	0	0	0	0	2	−2
Moment of momentum	*mva*	2	1	−1	0	2	1	−1	0	0	0	0	0	0	3	−2
Force......	....	1	1	−2	0	1	1	−2	0	0	0	0	0	0	2	−3
Energy....	mv^2	2	1	−2	0	2	1	−2	0	0	0	0	0	0	3	−3
Electric capacity	*C*	1	0	0	1	−1	0	2	−1	2	0	−2	1	1	0	1
Magnetic self-induction .	*L*	−1	0	2	−1	1	0	0	0	−2	0	2	−1	−1	0	1
Electric resistance	*R*	−1	0	1	−1	1	0	−1	1	−2	0	2	−1	−1	0	0
Electromotive force....	*E*	$\frac{1}{2}$	$\frac{1}{2}$	−1	$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	−2	$\frac{1}{2}$	−1	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	−2
Electric current..	*I*	$\frac{3}{2}$	$\frac{1}{2}$	−2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	−1	$-\frac{1}{2}$	1	0	−1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	−2
Magnetomotive force....	....	$\frac{3}{2}$	$\frac{1}{2}$	−2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	−1	$-\frac{1}{2}$	1	0	−1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	−2
Electric force....	....	$-\frac{1}{2}$	$\frac{1}{2}$	−1	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	−2	$\frac{1}{2}$	−1	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	−2
Magnetic force....	*H*	$\frac{1}{2}$	$\frac{1}{2}$	−2	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	−1	$-\frac{1}{2}$	1	0	−1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	−2
Electric displacement....	*D*	$-\frac{1}{2}$	$\frac{1}{2}$	−1	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	0	−1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	−1
Magnetic flux density or induction .	*B*	$-\frac{3}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	−1	$\frac{1}{2}$	−1	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	−1
Electric quantity	*Q*	$\frac{3}{2}$	$\frac{1}{2}$	−1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	0	−1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	−1
Total magnetic flux	*Z*	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	−1	$\frac{1}{2}$	−1	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	−1
Quantity magnetism.....	....	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	−1	$\frac{1}{2}$	−1	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	−1
Planck's constant	*h*	2	1	−1	0	2	1	−1	0	0	0	0	0	0	3	−2
Rydberg's constant	*K*	0	0	−1	0	0	0	−1	0	0	0	0	0	0	0	−1
Newtonian constant	*k*	3	−1	−2	0	3	−1	−2	0	0	0	0	0	0	2	−1

the same dimensions in the space-time system, indicating that they are of the same nature. And, again, electric capacity and the coefficient of self- or mutual induction

receive the same dimensions, that of a time. Electromotive force receives the same dimensions as magnetomotive force. Electric resistance comes out dimensionless, being the ratio between two velocities like the quantity β, which is the ratio of the velocity of an electron to the velocity of light. This fact gives electric current the same dimensions as electromotive force because of the relation in Ohm's law, $R = E/I$. And, again, electric force and magnetic force have the same dimensions. So, also, do electric flux, or displacement, and magnetic flux density come out of the same dimensions. Energy comes out as the cube of a velocity, which becomes rational when we regard energy as equivalent to mv^2 and mass as a velocity.

It may at present be difficult to obtain any mental picture from these dimensional formulæ for the different kinds of quantities, but it is maintained that it was still more difficult when the dimensions of these quantities were expressed in two independent systems, the electrostatic and the electromagnetic. Without having some mental picture of the quantities k and μ, it was impossible to harmonize the very different-looking dimensions of the very same quantity as expressed in the two systems. This table of dimensions makes an appeal to reason in such a strong way that it is regarded as strong support for the ideas that have led to the determination of the dimensions of mass, specific inductive capacity, and magnetic permeability, that is to say, in support of the theories here advanced.

XIV

LET us next return to the equation derived from electromagnetic theory but modified by the factor $x\beta_1^2$, or $m_0\beta_1^2$ in (201). The presence of this factor makes a tremendous difference in the magnitude of the force, since m_0 itself is about $.9 \times 10^{-27}$ and the β_1^2 of the order of 10^{-4}. Let us in advance of the matters which follow admit that there is very great probability, on account of the results obtained, that the original form of electromagnetic theory will have to be changed in some way to fit the case. Just what the change in the original equations will be is difficult to say on this evidence alone, but an attempt[1] has been made in the original article in which this equation was derived to trace back to their origin the particular terms in the fundamental expression of electromagnetic theory that give rise to the force varying as the inverse square of the distance. For there are many other terms in the original equation which have no effect at all so far as the inverse square of the distance terms are concerned. It was there pointed out that all of the inverse square of the distance terms in this so-called gravitational equation arise either from the differentiation of the scalar or vector potential, ϕ or α, with respect to the time in distinction to the space coördinates.

This matter seems to be very significant now in the light of the recent papers by Saha above referred to. For he has shown that in the so-called Doppler factor,

[1] *Loc. cit., Phys. Rev.*, June, 1917, p. 464.

(197) above, the $\partial\tau$ should be replaced by a generalized value depending equally upon the four coördinates in the generalized Minkowski space, which includes x, y, and z as well as t. This change will of necessity make a difference in the value of the Doppler factor. At the present time it is not possible to say what it will become, but the point is that there exist to-day good grounds for supposing that it is different from the expression already given in (197), in which the differentiation is with respect to the time only. If we suppose that a similar factor having a small value, such as $m_0\beta_1^2$ should be introduced into the second term of the Doppler factor, then we are entirely justified in considering that the factor is sensibly equal to unity, as it was assumed to be in the derivation of the gravitational equation (196). Also the work of Schott above referred to, which does not lead to a correct gravitational form, cannot be a correct result on the changed premises, namely that the Doppler factor has undergone a change. It should be an argument in favor of a revision of this Doppler factor that the results obtained regarding it as sensibly equal to unity agree in every respect with the gravitational law, after we have introduced the factor, $m_0\beta_1^2$. But legitimate grounds have appeared through the work of Saha to modify the form of the Doppler factor independently of any other considerations.

Let us now apply the equation to find the attraction between two material bodies at a great distance apart. As it stands, the equation represents the force between a single pair of electrons, one in each of the two bodies. The only quantities that have any different values in the equation when applied to a second pair of electrons are the velocities, β_1 and β_2, the angle, α, and direction cosines, X and Z, which occur within the bracket. The

distance, r, may be regarded as so great in comparison with the size of the body that r is very approximately the same for any two pairs of electrons in the bodies, taking one from each body.

It is evident that the force will be different between the same pair of electrons having the same radius of orbit and same angular velocity, depending upon how the planes of the orbits are turned with respect to each other. In any body save a crystal the chances are that there will be orbits turned in every possible manner with respect to each other, and it therefore becomes desirable to find the average force between a single pair of electrons as they are turned in every conceivable manner without changing the positions of the centers of the orbits. We have already taken a time average of the force as the electrons proceed around their orbits, and now we want a space average of the force as the orbits themselves are turned in all possible ways. The process of obtaining this space average is given in Appendix A so as not to divert attention from the main argument at present, and the result of it is that the space average of the force is obtained by simply replacing the bracket in equation (201) by the numeric $\frac{2}{3}$, giving the result as follows,

$$F_r = \tfrac{1}{3}e^2 x \beta_1^2 \beta_2^2 r^{-2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (205)$$

If, therefore, we write down the force of attraction between just one electron in the body, 1, and each of the electrons in the second body, 2, and add them all together, we would have an equation just like (205) in which the $\beta^2{}_2$ is replaced by $\sum\limits_{2}\beta^2$ taken over every electron in body 2. And, again, if we write down the attraction of the whole body, 2, for each electron in the body, 1, the total is expressed by an equation like (205) in which β_1^2 is replaced by $\sum\limits_{1}\beta^2$, the summations being extended

over all of the electrons in each of the bodies. This equation is as follows,

$$F = \tfrac{1}{3}\, e^2 x \sum_{1} \beta^2 \sum_{2} \beta^2 r^{-2}. \quad . \quad . \quad . \quad . \quad . \quad (206)$$

No further progress can be made toward getting a numerical value of the attraction without possessing some knowledge of the velocities of the electrons in the rings of electrons in the various atoms that enter into the material body. Use may now be made of the formula developed above for the velocity of the electrons in any ring of electrons in (172). Squaring this, we obtain β^2 for a single electron in the ring, and multiplying by the number of electrons in the ring, p, we find the sum of the squares of β for a ring of electrons, namely

$$\sum_{p} \beta^2 = \frac{p^2}{4} \frac{hm_H}{e^2 m_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (207)$$

To simplify matters as much as possible, let us consider the attraction between two atoms of the simplest kind, each atom having but a single ring of electrons as in hydrogen or helium. If the two bodies, 1 and 2, are alike, each being an atom of hydrogen, say, then the total value of the $\Sigma\beta^2$ for each body is just the expression in (207), and the product $\sum_{1}\beta^2\sum_{2}\beta^2$ is simply the square of (207). Substituting this in (206) gives the complete force between the two atoms of hydrogen as

$$F = \tfrac{1}{3} x \frac{p^4}{16} \frac{h^2 m^2_H}{e^2 m_0^2} r^{-2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (208)$$

This should represent the attraction on the average between the two hydrogen atoms or two helium atoms when the correct numbers of electrons in the rings are substituted for p in the two cases respectively.

But the attraction on the average between two atoms of hydrogen is given by Newton's gravitational law to be

$$F = km_H^2 r^{-2}. \qquad (209)$$

By equating the two expressions for the same force, (208) and (209), we may find a value for the gravitational constant, k, and thus see that the two expressions are really equivalent. We have

$$k = \frac{x}{3}\frac{p^4}{16}\left(\frac{h}{em_0}\right)^2, \qquad (210)$$

which gives a value for the Newtonian constant in terms of the properties of the electrons with the exception of the unknown quantities x and p, the latter being the number of electrons in the hydrogen atom. We shall take this number of electrons in the hydrogen atom in its normal condition as equal to 2 and solve the equation for x, giving

$$x = 3k\left(\frac{em}{h}\right)^2 \qquad (211)$$

Using the current values of e, m_0 and h, namely

$$e = 4.774 \times 10^{-10}$$
$$m_0 = .90 \times 10^{-27}$$
$$h = 6.547 \times 10^{-27}$$

and taking the Newtonian constant as equal to 666×10^{-10}, the value of x becomes

$$x = .8625 \times 10^{-27}. \qquad (212)$$

This numerical value is so close to the value of the mass of the electron in view of the uncertainties in the experimental values used in determining it that there are good grounds for thinking that the multiplier, x, should have been the very simple physical quantity, m_0, the mass

of the electron. It should be pointed out that there is some uncertainty in the value of the Newtonian constant. Astronomers usually take the mass of the earth as unity, but to obtain the constant on the C.G.S. system of units really involves the mass of the earth in grams, and it is difficult to say with assurance how much error there is in the above numerical value of k. The values of e and of h used above are those determined by Millikan. Substituting m_0 for x in (211) we obtain the expression for the gravitational constant

$$k = \frac{1}{3}\frac{h^2}{e^2m_0} \quad \text{. . . . . (213)}$$

The dimensions of this expression are correct. The dimensions of h are L^2MT^{-1}, and of h^2/e^2m_0 in the electrostatic system, Lk^{-1}, and in the space-time system, L^2T^{-1}, which is equivalent to those of the Newtonian constant.

The Newtonian equation of gravitation is

$$F = km_1m_2r^{-2},$$

whence $$k = F(m_1m_2)^{-1}r^2.$$

The dimensions of force are LMT^{-2}, and, according to this value of k, the dimensions of it are $L^3M^{-1}T^{-2}$. Replacing M^{-1} by its equivalent, $L^{-1}T$, we have the dimensions of k, L^2T^{-1}, which agrees with (213) above. Or we might have replaced the M^{-1} by its equivalent, k, the specific inductive capacity, and obtained $L^3T^{-2}k$, which becomes equivalent to the dimensions on the electrostatic system obtained above, namely Lk^{-1}, remembering that L^2T^{-2} is equivalent to k^{-2}. Although we have employed the same symbol, k, for both the Newtonian constant and for the specific inductive capacity in this work it is difficult to see how any confusion can arise from this.

The expression obtained in an article previously published[1] for the Newtonian constant was a much more complicated one than that given above, namely

$$k = \frac{16 m_0 \pi^4 e^{10}}{3 k' m_H{}^2 c^4 h^4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (214)$$

This was obtained through the use of the Bohr value of the Rydberg constant, which gives a numerical value in close agreement with the above. The quantities π^4, c^4 and $m_H{}^2$ occur in this in addition to the quantities that occur in the simpler expression (213). The specific inductive capacity is denoted by k' here, since the two k's occur in this equation. The powers of the quantities are very high, that of e being the tenth power. The simplicity of the value in (213) is greatly in its favor. Moreover, the dimensions of this expression do not agree with those of the Newtonian constant, even if we take the specific inductive capacity as the reciprocal of a mass. This fact strengthens the grounds for the opinion that the Bohr expression for the Rydberg constant does not represent a true equation between physical quantities when the k is omitted.

Denoting by m_1 and m_2 the masses of two bodies, their attraction for each other is by Newton's law,

$$F = k m_1 m_2 r^{-2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (215)$$

Substituting the value of k in (213) above, gives

$$F = \frac{1}{3} \frac{h^2}{e^2 m_0} m_1 m_2 r^{-2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (216)$$

And by (206) we have

$$F = \tfrac{1}{3} e^2 m_0 \sum_1 \beta^2 \sum_2 \beta^2 r^{-2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (217)$$

[1] *Loc. cit., Phys. Rev.*, July, 1918, p. 15, equation (9).

Equating (216) and (217) gives

$$m_1 m_2 = \frac{e^4 m_0^2}{h^2} \sum_1 \beta^2 \sum_2 \beta^2, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (218)$$

from which we derive the value of m_1 or m_2, namely

$$m = \frac{e^2 m_0}{h} \Sigma \beta^2. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (219)$$

The dimensions of this expression for the mass of a body in general are correct, for $\Sigma\beta^2$ has no dimensions, and on the electrostatic system e^2/h has the dimensions $LT^{-1}k$, and putting the k equal to $L^{-1}T$, the dimensions of e^2/h become zero on the space-time system, thus reducing the expression on the right of (219) to the dimensions of mass. This tells us that the mass of a body is proportional to the sum of the squares of the velocities of all the electrons in the body, that is, proportional to the kinetic energy of all the electrons within the body.

The expression for the mass of a body as given in (219) may seem surprising at first, since we attribute the mass of a body to the sum of the masses of the nuclei of all the atoms in the body, and the nucleus makes no appearance in this expression. It should be remembered that the masses of bodies are strictly proportional to their weights, and the expression in (219) is derived from the weight of the body primarily. There is great probability that the expression for the mass derived from summing up the masses of the separate nuclei of the atoms in a body will come out equivalent to the expression in (219) because the sum of the electrical charges is the same for the nuclei as for the total number of electrons. The Lorentz mass formula for one atomic nucleus is

$$m = \frac{4}{5} \frac{E^2}{c^2 ak},$$

where E denotes the positive charge of the nucleus. If the number of atoms in the body in question is A, many of the atoms being of different kinds so that the radii, a, differ, the total mass of the body is

$$M = \sum_A m = \frac{4}{5c^2k}\sum_A \frac{E^2}{a}.$$

. . . .

There is no reason to suppose that any different value would be obtained from this summation than that given above in (219), although the summation may be difficult to obtain except in special cases. If the formula is applied to the simple case of hydrogen, where the atoms are all alike, and where $E = 2e$, and where the radius, a, is the same in every atom, namely, according to (152) and (148) above

$$a = \frac{8}{5Kk} = 3.2\frac{e^2}{m_H c^2 k} = 4.8620 \times 10^{-16}, \quad . \quad . \quad . \quad . \quad (153)$$

we have the mass of the hydrogen gas

$$M = \sum_A m_H = \frac{4}{5c^2k}\sum_A \frac{(2e)^2}{a} = \frac{4}{5c^2k} \times \frac{4m_H c^2 kA}{3.2} = m_H A,$$

which is evidently equal to the mass of the gas because it is the mass of one atom times the number of atoms.

The formula (219) gives an equal result, for by (207) the sum of β^2 for one atom is hm_H/e^2m_0, and for A atoms is A times this. When this value is substituted in (219), the coefficient, e^2m_0/h cancels, leaving only m_HA for the mass of the gas, the same expression as obtained from the nucleus of the atom.

It is of interest to note that we have just pointed out that e^2 and h have the same dimensions in the space-time system, and that each is the same as a moment of momentum, L^3T^{-2}. From this it seems likely that h is closely connected with e and is constant because e is

constant. And again it may be conjectured that the energy content of the negative electron and the positive hydrogen nucleus are as follows, for we may write the Lorentz mass equation

$$m_H c^2 = \frac{16e^2}{5a_H k} = 1.658 \times 10^{-24} \times 9 \times 10^{20} = 1.492 \times 10^{-3},$$

$$m_0 c^2 = \frac{4e^2}{5a_0 k} = .898 \times 10^{-27} \times 9 \times 10^{20} = 8.082 \times 10^{-7}.$$

The dimensions of each of the members of these equations are those of energy, or the cube of a velocity, L^3T^{-3}, and the expression e^2/ak on the right of the equations is of the form of potential energy. We may think of these figures as representing the energy contained in the hydrogen nucleus, 1.492×10^{-3} ergs, and the electron itself, 8.082×10^{-7} ergs, the former being about 1845 times the latter. As compared with the energy, hK, which represents the order of magnitude of kinetic energy of the electrons in their orbits, namely $.2154 \times 10^{-10}$ ergs, the energy content of the electron itself is very large, 37,500 times greater.

XV

IF the gravitational equation (217) is applied to the earth as one of the bodies and to a second body on the surface of the earth, this equation then expresses the weight of that body. When different bodies are substituted for the one, the earth remaining as the common body, it is evident that the weight will vary as the sum of the squares of the velocities of all the electrons in the body, and hence the weight is proportional to the mass as it is known to be in fact.

The equation enables us to write down the weight of any kind of a body small or large, and it shows that the weight contributed to any single atom by one of the rings of electrons in it depends merely upon the sum of the squares of the velocities of the electrons in the ring. We have made the velocity of the electrons in any ring depend only upon the number of electrons in the ring in equation (172) above, no matter how many other rings of electrons there may happen to be in the same atom, but this is regarded only as a first approximation, the velocity also depending upon the radius of the orbit so far as second order terms are concerned. Let us then apply the gravitational equation to write down the weights of rings of electrons only. Now one body is the earth and the other body is a single ring of electrons on its surface. The average weight of the ring of p electrons when oriented in all possible ways is then

$$F = \tfrac{1}{3}e^2 m_0 \sum_p \beta^2 \sum_E \beta^2 r_E^{-2}, \quad . \quad . \quad . \quad . \quad . \quad (220)$$

where $\sum_{p}\beta^2$ refers to the ring of p electrons only and $\sum_{E}\beta^2$ refers to all the electrons composing the mass of the earth, r_E becoming the radius of the earth. If, therefore, we compare the weights of two rings of electrons having different numbers in the rings, and use the expression

$$\sum_{p}\beta^2 = \frac{p^2}{4}\frac{hm_H}{e^2 m_0}$$

given in (207) above for the ring, it is apparent that the only variable quantity that changes when one ring is substituted for another is the p^2, for the other quantities in this expression are constants, and, of course, the unknown quantities pertaining to the earth, the other body, do not change. It may be stated, therefore, that the weights of rings of electrons are proportional to the squares of the numbers of electrons they contain. The absolute weights may then be written down as soon as the weight of some one ring is known. Let us say that the weight of a ring of two electrons is the same as the weight of a hydrogen atom, and take it as 1.008 on the scale, making the weight of the oxygen atom equal to 16, which is customary. The weights of rings of electrons of 3, 4, etc., on this scale will be

p electrons per ring	Weight of ring
2	1.008
3	2.268
4	4.032
5	6.300
6	9.072

. . . (221)

Founded upon this suggestion an attempt has been made to ascertain from the known atomic weights of the

Table Calculated with the Numbers in (222)

Element	At. wt. observed O=16 H=1.008	At. wt. calculated O=15.9990 H=1.008	Total no. of electrons	Arrangement in rings, in atoms					Per cent error in measured at. wt.	Per cent difference in calculated at. wt.
				2	3	4	5	6		
H	1.008	1.008	2	1	...	...	...	...	0.0993	0.000
He	4.00	3.99975	4	...	...	1	...	...	0.25	− 0.00616
Li	6.94	6.8893	9	...	3	...	...	...	0.144	− 0.731
Gl	9.1	9.1371	6	...	...	...	...	1	1.10	0.408
B	11.0	11.0235	14	3	...	2	...	...	0.909	0.214
C	12.00	11.99926	12	...	...	3	...	...	0.0833	− 0.00616
N	14.01	14.01587	16	2	...	3	...	...	0.0714	0.0376
O	16.00	15.9990	16	...	...	4	...	...	0.0625	− 0.00616
F	19.0	19.023	22	3	...	4	...	...	0.526	0.121
Ne	20.2	20.3115	23	2	1	4	...	...	0.495	0.552
Na	23.00	23.0228	26	3	...	5	...	...	0.0435	0.099
Mg	24.32	24.3112	27	2	1	5	...	...	0.0413	− 0.0362
Al	27.1	27.0225	30	3	...	6	...	...	0.369	− 0.286
Si	28.3	28.3109	31	2	1	6	...	...	0.3533	0.0387
P	31.04	31.0223	34	3	...	7	...	...	0.0322	− 0.0571
S	32.06	32.0625	40	8	...	6	...	...	0.0312	0.00793
Cl	35.46	35.4488	29	...	...	1	5	...	0.0282	− 0.0315
A	39.88	39.8953	43	1	3	8	...	...	0.0251	0.0384
K	39.10	39.054	46	7	...	8	...	...	0.0256	− 0.1175
Ca	40.07	40.062	48	8	...	8	...	...	0.02495	− 0.0199
Sc	44.1	44.062	52	8	...	9	...	...	0.227	− 0.0867
Ti	48.1	48.062	56	8	...	10	...	...	0.208	− 0.0800
V	51.0	51.053	58	7	...	11	...	...	0.196	0.1045
Cr	52.0	52.061	60	8	...	11	...	...	0.192	0.1178
Mn	54.93	54.919	61	4	3	11	...	...	0.0182	− 0.0208
Fe	55.84	55.831	38	1	...	...	...	6	0.0179	− 0.0165
Co	58.97	59.020	62	3	...	14	...	...	0.0169	0.0857
Ni	58.68	58.638	66	6	2	12	...	...	0.01705	− 0.0717
Cu	63.57	63.613	68	3	2	14	...	...	0.01573	0.0683
Zn	65.37	65.317	69	3	1	15	...	...	0.0153	− 0.0814
Ga	69.9	70.012	72	2	...	17	...	...	0.1431	0.1599
Ge	72.5	72.589	74	...	2	17	...	...	0.1379	0.1223
As	74.96	75.020	78	3	...	18	...	...	0.0133	0.0795
Se	79.2	79.185	80	2	2	15	2	...	0.1262	− 0.0192
Br	79.92	79.905	68	1	...	4	10	...	0.0125	− 0.0185
Kr	82.92	82.917	89	4	3	18	...	...	0.01206	− 0.00378
Rb	85.45	85.494	91	2	5	18	...	...	0.01169	0.0512
Sr	87.63	87.612	92	3	2	20	...	...	0.01141	− 0.0206

Element	At. wt. observed O=16 H=1.008	At. wt. calculated O=15.9990 H=1.008	Total no. of electrons	Arrangement in rings, in atoms.					Per cent error in measured at. wt.	Per cent difference in calculated at. wt.
				2	3	4	5	6		
Yt	88.7	88.900	93	2	3	20	...	...	0.1127	0.226
Zr	90.6	90.604	94	2	2	21	...	...	0.1104	0.00407
Cb	93.1	93.181	96	...	4	21	...	...	0.1074	0.0866
Mo	96.0	96.026	100	4	...	23	...	...	0.1042	0.0275
Ru	101.7	101.595	104	1	2	24	...	...	0.0983	−0.1033
Rh	102.9	102.883	105	...	3	24	...	...	0.0972	−0.0162
Pd	106.7	106.603	110	2	2	25	...	...	0.09375	−0.0912
Ag	107.88	107.891	111	1	3	25	...	...	0.00927	0.0103
Cd	112.40	112.306	115	2	1	27	...	...	0.00889	−0.0838
In	114.8	114.883	117	...	3	27	...	...	0.0871	0.0720
Sn	118.7	118.602	122	2	2	28	...	...	0.0842	−0.0826
Sb	120.2	120.305	123	2	1	29	...	...	0.0832	0.0876
Te	127.5	127.456	125	...	2	26	3	...	0.0784	−0.0346
I	126.92	126.938	109	...	2	7	15	...	0.00788	0.0145
Xe	130.2	130.288	131	...	1	32	...	...	0.0768	0.0680
Cs	132.81	132.795	140	3	6	29	...	...	0.00753	−0.0110
Ba	137.37	137.312	141	3	1	33	...	...	0.00728	−0.0420
La	139.0	139.015	142	3		34			0.0719	0.0112
				Rare Earths						
Ta	181.5	181.590	184	1	2	44	...	...	0.0551	0.0496
W	184.0	184.021	188	4	...	45	...	...	0.05435	0.0116
Os	190.9	190.878	193	...	3	46	...	...	0.05238	−0.0115
Ir	193.1	193.174	196	...	4	46	...	...	0.0518	0.0385
Pt	195.2	195.293	197	1	1	48	...	...	0.0512	0.0474
Au	197.2	197.308	201	3	1	48	...	...	0.0507	0.0550
Hg	200.6	200.613	206	4	2	48	...	...	0.04985	0.00650
Tl	204.0	204.020	208	4	...	50	...	...	0.0490	0.00965
Pb	207.20	207.292	209	1	1	51	...	...	0.00483	0.0443
Bi	208.0	208.052	216	8	...	50	...	...	0.04806	0.0249
Nt	222.4	222.315	227	4	1	54	...	...	0.04495	−0.0381
Ra	226.0	226.034	232	6	...	55	...	...	0.0442	0.0152
Th	232.4	232.331	239	6	1	56	...	...	0.0430	−0.0298
Ur	238.2	238.282	239	...	1	59	...	...	0.0420	0.0344

different kinds of atoms the particular combination of rings of electrons, each having these approximate weights, as given in the table, which will add up to make the known weight of the atom. After many trials it has been found that, by slightly altering the numbers in the table according to the following values, a very large majority of atomic weights may be obtained with accuracy, namely

p electrons per ring	Weight of ring
2	1.008
3	2.29643
4	3.99975
5	6.2895
6	9.137

. . . (222)

It will make the process of calculation clearer if an example is given. The element magnesium is given in the table as having eight rings of electrons total made up of five rings of four, one ring of three, and two rings of two electrons, or a total of 27 electrons. The weights supposed to be contributed to the atom by the rings are as follows:

$$\begin{array}{lcl} 5 \text{ rings of four} = 5 \times 3.99975 & = & 19.99875 \\ 1 \text{ ring of three} = 1 \times 2.29643 & = & 2.29643 \\ 2 \text{ rings of two} = 2 \times 1.008 & = & \underline{2.016} \\ \quad \text{Total} \ldots\ldots\ldots & & 24.3112 \quad \ldots\ldots (224) \end{array}$$

The measured atomic weight of magnesium is 24.32, and admitting that it is possible that an error of at least one unit in the last decimal place in the experimental weight has been made, the calculated value comes within the experimental error.

It may be urged as an objection to this scheme that the number of possible combinations which can be made

of these numbers that will approximate to any desired number is very great. This objection has some force, but it will not apply to the elements of low atomic weight, and it is here that a real test of the scheme is obtained. The calculation has been extended to the heavier elements largely because the first part of the table of low atomic weights indicates the nature of the whole scheme as being largely based upon a preponderating number of rings of four electrons, and scattering numbers of rings of three and of two. Rings of five and six are seldom required, and no ring greater than six appears. Rings of five appear in the halogens, chlorine having five, bromine ten, and iodine fifteen.

It is not contended that the above table gives the correct number of electrons in the atoms in every case, and will not be subject to future revision, but the main idea which runs through the table is that most of the atoms are made up of rings of four electrons. Nothing is intimated as to the order of succession of the rings as we proceed out from the nucleus, that is, whether the rings of two are within or outside of the rings of three or four. If we take one atom of each kind as given in the table and add up the number of rings of electrons to find a total, we shall find that there are in the 70 atoms

1470 rings of four electrons
185 rings of two electrons
86 rings of three electrons
35 rings of five electrons
7 rings of six electrons . . . . . (225)

thus indicating that the rings of four electrons greatly predominate in any mass of matter made up of a mixture of elements, such as the earth for example.

This feature which is essential to the scheme may

be tested by means of the gravitational equation, as will now be explained, and it is proved to be in complete harmony with the gravitational equation.

As a first illustration, let us apply the formula to obtain the attraction between the earth and a single hydrogen atom upon its surface. We have, by Newton's law,

$$F = km_H m_E r_E^{-2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (226)$$

and, by the gravitational equation,

$$F = \tfrac{1}{3}e^2 m_0 \sum_H \beta^2 \sum_E \beta^2 r_E^{-2}. \quad . \quad . \quad . \quad . \quad . \quad (227)$$

Equating these two expressions for the same force, it is seen that the only unknown quantity is $\sum\limits_E \beta^2$ for the earth, which may, therefore, be found. It is

$$\sum_E \beta^2 = \frac{3km_H m_E}{e^2 m_0 \sum\limits_H \beta^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (228)$$

Substituting the following numerical values, namely

$$\begin{aligned} k &= 666 \times 10^{-10} \\ m_H &= 1.662 \times 10^{-24} \\ m_E &= 5.984 \times 10^{27} \\ e &= 4.774 \times 10^{-10} \\ m_0 &= .90 \times 10^{-27} \end{aligned}$$

$$\sum_H \beta^2 = \frac{bm_H}{e^2 m_0} (\text{see } (207)) = .531 \times 10^{-4}.$$

We obtain numerically

$$\sum_E \beta^2 = 1.825 \times 10^{47}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (229)$$

Use may be made of this sum of the squares of the velocities of all the electrons in the earth to find the average speed of a single electron in the earth by dividing this number by the total number of electrons in the

earth. Fortunately we know the approximate number of electrons in the earth as nearly as we know the mass of the earth in grams, for the number of electrons per gram of all substances except hydrogen is equal to the Avogadro constant, namely, 6.062×10^{23}. The total number of electrons in the earth is then approximately

$$N = 6.062 \times 10^{23} m_E = 3.6275 \times 10^{51}. \quad . \quad . \quad (230)$$

And now dividing $\sum_E \beta^2$ by N we obtain the mean square velocity of a single electron in the earth, as

$$\overline{\beta_E}^2 = .503 \times 10^{-4}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (231)$$

and taking the square root, the average velocity of an electron in the earth is

$$\overline{\beta}_E = .0071. \quad . \quad . \quad . \quad . \quad . \quad . \quad (232)$$

For comparison, we give the numerical values of the velocities of the electrons in rings of electrons calculated from the formula (170).

p electrons per ring	β = the velocity in terms of velocity of light
2	0.003641
3	0.005462
4	0.007283
5	0.009104
6	0.010925

. . (233)

It is seen from this table that the velocity of an electron in a ring of four electrons is very nearly equal to and a little greater than the average velocity of an electron in the earth, the comparison being .0071 and .00728. The average velocity for the earth is much nearer to that of a ring of four than to that of any other ring. This supports in a very striking way the scheme in the atomic

weight table above given and shows that the number of rings of four electrons in the atoms of the earth greatly preponderate, as it should according to the table.

Now it will be noticed that the mass of the earth occurs both in (228) and (230), so that when one is divided by the other this mass canceled. No error was introduced into the result, therefore, because of a wrong value for the mass of the earth. This leads to important considerations because it is apparent that for the earth might have been substituted any other body of mixed matter, the sun or any of the planets, and we would have obtained the same average velocity for a single electron. This does not seem strange because, according to the scheme of the atomic weight table, the average velocity should be nearly equal to that in a ring of four electrons. But let us look at the equations with more attention because of this general result.

Let the gravitational equation be applied to the hypothetical case of the attraction between a star or a mass of nebulous gas composed, first, entirely of hydrogen and a single hydrogen atom at a great distance away, somewhere outside of the star. If M denotes the mass of the star, the Newtonian law gives the attraction as

$$F = km_H M r^{-2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (234)$$

and it does not matter what value r has, provided the single hydrogen atom with mass m_H is situated in a fixed position a long distance from the star. The general expression for mass is given in (219) above. and the ratio of these two masses is, therefore,

$$M/m_H = \sum_M \beta^2 / \sum_H \beta^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (235)$$

But the sum of β_H^2 for a single hydrogen atom is twice the square of the velocity of just one of the electrons

because there are two electrons, and we will denote this velocity by β_H. The last equation is then equivalent to

$$\beta_H^2 = \frac{\sum_M \beta^2}{\frac{2}{m_H} M}. \quad \ldots \ldots \ldots \quad (236)$$

Now the average speed of the electrons in the star must be the same as a single electron in an hydrogen atom, β_H, because we have made the hypothesis that the star is entirely composed of hydrogen atoms. And since the numerator on the right of the equation represents the sum of the squares of the velocities of all the electrons in the star, and the quotient, β_H^2, the left member of the equation, represents the average square of the velocities, it must be that the denominator on the right represents the total number of the electrons in the star, say N,

$$N = \frac{2}{m_H} M. \quad \ldots \ldots \ldots \quad (237)$$

And since M is the number of grams mass of the star, then $2/m_H$ must be the number of electrons in one gram mass of hydrogen. This is evidently true because m_H represents the mass of one hydrogen atom, and the number of such atoms in one gram must be $1/m_H$. The number of electrons per gram is twice this quantity, or $2/m_H$. The reciprocal of the mass of the hydrogen atom $1/m_H$ is very nearly equal to the Avogadro constant, and we may say, then, that the number of electrons in a gram of hydrogen is about twice the Avogadro constant. The number of electrons in any other element, however, is nearly equal to the Avogadro constant, and not twice it, as we shall see, hydrogen being an exception, as it is in several other particulars.

Let us now take another example and suppose that

the star is entirely composed of helium instead of hydrogen, and assume that the helium atom has a single ring of four electrons. The ratio of the masses of the star, M, to the single helium atom, m_{He} is then

$$M/m_{He} = \sum_{M}\beta^2 / \sum_{He}\beta^2. \quad . \quad . \quad . \quad . \quad . \quad (238)$$

Denoting the speed of one electron in the helium atom by β_{He}, then

$$\sum_{He}\beta^2 = 4\beta^2{}_{He} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (239)$$

and (238) becomes equivalent to

$$\beta_{He}{}^2 = \frac{\sum_{M}\beta^2}{\frac{4}{m_{He}}M} \quad . \quad . \quad . \quad . \quad . \quad . \quad (240)$$

In a similar manner to the case of the hydrogen star, the numerator on the right represents the sum of the squares of the velocities of all the electrons in the star, and the quotient on the left represents the average square of the velocity of one electron in the star. Hence the denominator on the right must represent the total number of electrons in the star, say N,

$$N = \frac{4}{m_{He}}M. \quad . \quad . \quad . \quad . \quad . \quad . \quad (241)$$

Hence $4/m_{He}$ must be the number of electrons per gram of helium. And evidently $1/m_{He}$ is the number of helium atoms in one gram, and four times this is the number of electrons in the gram.

The mass of the helium atom in terms of that of hydrogen is

$$m_{He} = \frac{4.00}{1.008}m_H = 3.9683m_H \quad . \quad . \quad . \quad . \quad (242)$$

and

$$4/m_{He} = 1.008/m_H \quad . \quad . \quad . \quad . \quad . \quad . \quad (243)$$

Using the Millikan value of $m_H = 1.662 \times 10^{-24}$, we obtain

$$4/m_{He} = 6.065 \times 10^{23}, \text{ the number of electrons in one gram of helium.} \quad . \quad . \quad . \quad (244)$$

This number is very approximately equal to the well-known Avogadro constant, which is given by Millikan as 6.062×10^{23}. In deriving his result Millikan used a little more accurate value of the atomic weight of hydrogen, namely 1.0077 instead of 1.008, which would account for the difference in the last decimal place of the Avogadro constant. It thus appears that the number of electrons per gram of helium is equal to the Avogadro constant, and not twice this number as in the case of hydrogen. The average velocity of the electrons in this helium star is also the same as the velocity in a ring of four electrons, and approximately the same as the average velocity of the electrons in any other piece of matter made up of a mixture of different kinds of atoms.

This is a direct result of the fact that the number of electrons per gram of other substances than hydrogen is very nearly constant and equal to the Avogadro constant, and the reason for this is to be found in the scheme of the atomic weight table, which makes the rings of four electrons greatly preponderate over the other kinds of rings. It may be proved that the number of electrons per gram of all substances other than hydrogen is approximately constant, whether we assume that the number of electrons per atom is proportional either to the atomic number or to the atomic weight, it does not matter which. But instead of digressing here to give this proof, it is added in Appendix B. There has been considerable controversy over the question of the number of electrons in the various kinds of atoms, and some have made the number roughly proportional to the atomic number,

others to the atomic weight. Admitting that the hydrogen atom has two instead of one electron reconciles these two points of view, for the ratio of the number of electrons in an atom to the number in hydrogen may be considered to be very near to the atomic number, whereas the actual number may be nearer to the atomic weight, which is roughly twice the atomic number. There is some ground, therefore, for both contentions, or at least it is possible to see how both views have been held.

It is probable that one of the principal objections that will be raised to the atomic weight table (223) will be that the number of electrons in the various atoms is not made equal to the atomic number. It is the common idea among writers on these subjects to think of the number of positive unit charges on the nucleus as equal to the atomic number, that of hydrogen being unity, helium two, etc., exactly according to the atomic number. This conception has come about through the work of Moseley, who first obtained a linear relation between the different atoms by means of the X-ray spectra, and it was natural to infer that the uniform progression of the X-ray frequencies is connected with the fact that the charge on the electron has a fixed value and that their number increases by unity from element to element.

In reply to this it may be pointed out that this is still an inference not supported by any conclusive proof. The present theory seems to throw some light on this matter, for it has been rendered probable that the spectra of atoms is due primarily to the motion of the electrons connected with it, and we should look to this for regularities in the X-ray or the light spectra rather than directly to the nucleus. The quantity, *h*, Planck's constant, plays a part in this, and we have in this constant just as truly a fixed value as the value of the electrical

charge, and which might as easily be responsible for a regular increase in the spectra from element to element as electrical charge.

One might be tempted to suppose that the mass of the nucleus should also proceed by regular steps if its electrical charge increases by equal steps, but it does not. It may be worth while to see how the mass of the iron atom, for example, can be so great as to have atomic weight 55.84 and yet have the comparatively small number of electrons allotted to it in the table (38) in the light of this theory. The mass of the nucleus depends as well upon its radius as it does upon its electrical charge, according to the Lorentz mass formula. In the case of hydrogen we have seen that there is a direct relation between its radius and the Rydberg constant, namely

$$a_H = 8/5Kk \qquad \text{(See (161).)}$$

And this again is connected with the period of revolution of the electrons about the nucleus, the frequency being equal to $2K$, for the ring of two electrons. In the case of the iron atom, which we have made to consist of six rings of six and one ring of two electrons, the principal part of the frequency is that of a ring of six electrons instead of two. It seems legitimate to say that the radius of the iron nucleus is connected with the Rydberg constant by an analogous expression to that in (161), but that, the frequency of the rings of six being greater, we should have the radius, a, proportionately smaller than is the case in hydrogen. This would make the mass of the atom, which is inversely as its radius, proportionately larger than would correspond to the same number of electrons if they had the same speed as in hydrogen. In other words, it is easy to see by the aid of this theory that the mass of the iron atom may be 55.84, while the

number of electrons is exceptionally low, since the speeds or frequencies of revolution are exceptionally high. It is yet too early to give any exact calculation based upon the theory in the complicated case of iron of the radius of its nucleus. Its spectrum should first be reduced to a definite formula in a similar manner to that of hydrogen.

It appears to the author to be too simple a solution of an involved question to assume, as is commonly done, that the charge of the atomic nucleus increases by a fixed value from element to element in exact agreement with the atomic number.

APPENDIX A

It is proposed to show[1] that the force obtained from the gravitational equation (201) above is such that the bracket in that equation may be replaced by the numeric $\frac{2}{3}$ when the orbits of the two electrons take their average position or orientation with respect to each other in space, without, of course, altering the distance between their centers, r.

The quantities within the bracket, when expanded, become

$$1 - X^2 \sin^2 \alpha + 2XZ \sin \alpha \cos \alpha - Z^2 \cos^2 \alpha, \quad . \quad (1)$$

and we always have the relation between the direction cosines as follows,

$$X^2 + Y^2 + Z^2 = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

First, considering the center of the orbit of e_2 as fixed in position in space, the quantities r and z or Z are fixed, but not x and y, or X and Y, due to the way in which the axes have been defined. Let us now suppose that α is fixed and that the pole of the orbit of e_2 rotates around the elementary small circle of the sphere at a fixed latitude through an arbitrary angle, ϕ, from zero to 2π. It is easy to show from the definition of the axes that

$$\cos \phi = \frac{x}{(x^2 + y^2)^{\frac{1}{2}}} = \frac{x}{(r^2 - z^2)^{\frac{1}{2}}} = \frac{X}{(1 - Z^2)^{\frac{1}{2}}} \quad . \quad . \quad . \quad (3)$$

The value of the bracket (1) then becomes in terms of ϕ

$$1 - (1 - Z^2) \sin^2 \alpha \cos^2 \phi$$
$$+ 2 \sin \alpha \cos \alpha (1 - Z^2)^{\frac{1}{2}} Z \cos \phi - Z^2 \cos^2 \alpha. \quad . \quad . \quad . \quad . \quad (4)$$

[1] *Loc. cit., Phys. Rev.*, July, 1918, pp. 19, 20, and 21.

Regarding all quantities in this except ϕ as constant, we may obtain the average by integrating between zero and 2π. The average of $\cos^2\phi$ is $\frac{1}{2}$, and of $\cos\phi$ is zero between these limits, giving the result,

$$1 - \tfrac{1}{2}(1 - Z^2)\sin^2\alpha - Z^2\cos^2\alpha, \quad . \quad . \quad . \quad . \quad (5)$$

which is equivalent to

$$\tfrac{1}{2}(1 + Z^2) + \tfrac{1}{2}(1 - 3Z^2)\cos^2\alpha. \quad . \quad . \quad . \quad . \quad (6)$$

Let us next assume that Z remains fixed and average for α, thus obtaining the average over the first whole sphere. Again replace $\cos^2\alpha$ by its average $\frac{1}{2}$, giving

$$\tfrac{1}{2}(1 + Z^2) + \tfrac{1}{4}(1 - 3Z^2) = \tfrac{3}{4} - \tfrac{1}{4}Z^2. \quad . \quad . \quad . \quad (7)$$

And finally average for a change of Z between the limits zero and unity, as the center of the system of orbits, e_2, moves around the center of e_1 at the fixed radius, r, from the equator to the pole. The average of Z^2 between these limits is $\frac{1}{3}$, so that (7) becomes

$$\tfrac{3}{4} - \tfrac{1}{12} = \tfrac{2}{3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This completes the proof that the bracket in the gravitational equation (201) or (1) may be replaced by the numeric $\frac{2}{3}$ on the assumption that it is equally probable that the pole of the orbit of each electron e_1 and e_2 will lie in any one unit area of a sphere surrounding each center as in any other unit area of these spheres. This must be the case generally in all kinds of matter, — solids, liquids or gases, — crystals only excepted.

We will next show that this value $\frac{2}{3}$ will probably be obtained in the case of crystals also. To do this presupposes that the directions of the axes of rotation of the atoms in the crystal are known. In the cubic or isometric system of crystals, which is the only system

that the author has theoretically investigated[1] as yet, it has been proved that the directions of all the axes of rotation are equally divided into four groups, the axes in each group being all parallel to each other. The relative directions of these axes as between the four different groups are exactly according to the relative directions of the four medial lines of a regular tetrahedron. That is to say, one group has axes parallel to one of the medial lines of the tetrahedron, the second group to the second medial line, the third to the third, and the fourth to the fourth.

The proof[2] of this proposition is very simple and depends only upon the fact that each atom exerts a turning moment of force upon every other atom in the crystal tending to turn the plane of its orbit until it comes into parallelism with the given atom, when the turning moment vanishes. Without knowing or assuming any law governing these turning moments, if we merely assume that these moments are the same in similar positions of the atoms, then it may easily be shown that the sum of the turning moments of all the other atoms in the crystal acting upon any selected atom becomes zero and also produces stability, so that any displacement from this stable position brings into play a restoring couple, provided the directions of all the axes of rotation are grouped in the way that has been published elsewhere. This grouping makes the axes of all atoms take directions in four equal groups parallel to the four medial lines of a regular tetrahedron, as just stated.

To study the behavior of a cubic crystal, therefore,

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[1] *Loc. cit., Phil. Mag.*, June, 1915, p. 750, particularly p. 763, and Figure 1. A. C. Crehore, *Phil. Mag.*, Vol. XXX, August, 1915, p. 257.

[2] *Loc. cit., Phil. Mag.*, June, 1915, pp. 766, 767.

by means of the gravitational equation, it is only necessary to study the behavior of a group of four electrons, the orbits of which have their axes parallel to the four medial lines of a regular tetrahedron respectively. First, it may easily be shown that the quantity

$$r^2 - (-x \sin \alpha + z \cos \alpha)^2$$

is geometrically represented by the square of the perpendicular line from the center of the orbit of the electron, e_1, upon which we are getting the force, drawn to the axis of rotation of the electron e_2. Dividing this expression through by r^2 gives

$$1 - (-X \sin \alpha + Z \cos \alpha)^2,$$

which occurs within the bracket of equation (201). If, now, we write down the force according to (201) for each of the four electrons, e_2, having their axes parallel to the four medial lines of a regular tetrahedron respectively, and add together these four forces to obtain the effect of the group of four upon the one electron, e_1, we are in effect adding together the squares of the four perpendicular lines drawn from the center of e_1 upon each of the four axes of the orbits of e_2.

The sum of these forces will be a constant quantity, no matter in what direction the tetrahedron is turned, for it has been possible to establish the truth of the following geometrical theorem[1] upon which this matter depends. "If through any point four lines be drawn, making equal angles each with any other, and if from a second point at a fixed distance, r, from the first point four perpendiculars be drawn one to each of the said four lines, then the sum of the squares of these perpendiculars is constant for all points at the same distance

[1] *Loc. cit., Phys. Rev.*, June, 1917, p. 459.

from the first point. The locus of the second point is the surface of a sphere with the first point as center."

It can make no difference in the force upon the electron e_1, therefore, how the group of four electrons is oriented with respect to it. This is true of any other electron in the body of which e_1 is a selected electron. It is also true of a multitude of groups of four electrons, e_2, in the body 2, which may be supposed to be a cubic crystal. Hence we conclude that the gravitational equation shows that the attraction between two cubic crystals is strictly independent of their relative orientation, as it is known to be in fact.

It remains to obtain the value of the bracket in equation (201), and see whether it can be replaced by the factor $\frac{2}{3}$ as it was in the case of the general average obtained for all solids, liquids, and gases above. Since it makes no difference how the single group of four electrons is oriented, let us so place the medial lines of the tetrahedron that one of them passes directly through the centers of the orbits of e_1 and of e_2. The other three medial lines will then make equal angles with the line joining centers. The perpendicular distance from the center of the orbit of e_1 upon each of these other three medial lines of the tetrahedron is then equal to $\frac{2\sqrt{2}}{3}r$, and the sum of their squares is $\frac{8}{3}r^2$. Hence the sum of the four forces is given by replacing the bracket in (201) by the quantity $\frac{8}{3}$, and consequently the average force per pair of electrons as the group is oriented in all possible ways is $\frac{1}{4}$th of this, or the bracket must be replaced by the quantity $\frac{2}{3}$, which is precisely the same value that was obtained above for any other kind of substance.

In advance of the theoretical investigation of other

systems of crystals it cannot be said that we have established this proposition in general for all kinds of crystals. But it will certainly be a good guide in the study of other crystals first to assume that all axes are divided into four equal groups, each parallel to one of the medial lines of a regular tetrahedron, and see whether this proposition cannot be generally established.

APPENDIX B

Prove that the number of electrons in a gram[1] of substances in general, hydrogen excepted, is a constant quantity, if we start with the assumption that the number of electrons per atom is proportional either to the atomic number or to the atomic weight.

It is known that in a perfect gas, whether elemental or compound, the number of molecules per cubic centimeter is a constant quantity under standard conditions of pressure and temperature. This number is referred to as the gas-constant and may be denoted by N, say. If d is the density of the gas, the mass of a volume, V, of it is $m = Vd$. If we confine the attention to a volume of gas which has a mass of one gram, $m = 1$ and $V = 1/d$. The number of molecules in one cubic centimeter is N, and the molecules in one gram are, therefore, $NV = N/d$.

Let us now assume that there are p electrons in one molecule of the gas. Then the number of electrons per gram must be equal to

$$A = pNV = pN/d.$$

For two different gases it may be shown that $p/p' = d/d'$, or $p/d = p'/d'$, on the hypothesis that p is proportional to the atomic weight or to the atomic number, and hence $A = pN/d = p'N/d' = p''N/d''$ = a constant quantity.

It remains to prove that $p/d = p'/d' = p''/d''$, etc., for different gases. Suppose that the complex molecule of the gas is made up of n_1 atoms having atomic weight

[1] A. C. Crehore, *Phys. Rev.*, N. S., Vol. X, No. 5, October, 1917. See pp. 447, 448.

A_1, n_2 atoms having atomic weight A_2, etc., then the weight of the molecule, M, the molecular weight, is

$$M = n_1A_1 + n_2A_2 + n_3A_3 + \cdots \text{ etc.}$$

If the number of electrons per atom, P, is proportional to the atomic number or to the atomic weight, we have the atomic weights, A_1, A_2, etc., equal to some constant, say b, times their respective numbers of electrons, P_1, P_2, etc., that is

$$A_1 = bP_1; \ A_2 = bP_2, \text{ etc.}$$

Hence

$$M = b(n_1P_1 + n_2P_2 + \text{etc.}) = bp,$$

since the number of electrons per molecule, p, is equal to

$$p = n_1P_1 + n_2P_2 + \text{etc.}$$

Multiplying the molecular weight, M, by the number of molecules in one cubic centimeter of the gas, N, gives the mass contained in one cubic centimeter of it, as

$$NM = Nbp.$$

But the mass per cubic centimeter of a gas is the density of the gas by definition, hence

$$d = Nbp,$$

where N and b do not vary for different gases. For another gas this becomes

$$d' = Nbp',$$

whence $p/d = p'/d' = p''/d'' = \text{etc.} = 1/Nb$, a constant.

This establishes the proposition proposed, and makes the number of electrons per gram

$$A = pN/d = 1/b, \text{ a constant.}$$

The constant A may be considered as equal to the Avogadro constant, 6.062×10^{23}, and the constant b as the reciprocal of this.

APPENDIX C

GUIDED by the new space-time system of dimensions, it has been attempted to discover some function of the fundamental constants that will represent Planck's constant, h, both in numerical value and in dimensions, for there has appeared no place in the work above from which this important constant might be obtained except in the single instance of the Newtonian constant. It is considered that this is not known experimentally with an accuracy sufficient for our purposes. It will prove of interest to give the several forms of expressions for h thus found, which are equivalent. They are, first,

$$h = \left(\frac{a_H}{3}\right)^4 \frac{(2K)^3}{c}, \qquad \ldots\ldots\ldots \quad (1)$$

where a_H = the radius of the hydrogen nucleus, $2K$ is the frequency of revolution of the electrons in the normal hydrogen atom equal to twice the Rydberg constant, and c the velocity of light.

The dimensions of h must be those of energy multiplied by a time. On the electrostatic system energy has the dimensions $L^2MT^{-2}k^0$, and putting mass equal to a velocity, we obtain for energy L^3T^{-3}, or the cube of a velocity. Multiplying these by a time, the dimensions of h in the electrostatic system are $L^2MT^{-1}k^0$, and on the space-time system L^3T^{-2}.

The dimensions of the above expression for h are, therefore, in agreement with the required dimensions on the space-time system, for $a_H{}^4/c = L^3T$, and $K^3 = T^{-3}$, whence $a_H{}^4K^3/c = L^3T^{-2}$.

In (161) above we have given a value of the radius of the hydrogen nucleus in terms of the Rydberg constant, namely
$$a_H = 8/5Kk. \quad \ldots\ldots\ldots \quad (2)$$

By means of this we may eliminate a_H from (1) and find
$$h = 8^5/15^4k^4Kc. \quad \ldots\ldots \quad (3)$$

The dimensions of this expression for h are the same as those of (1) if we regard k as the reciprocal of a velocity, for $1/k^4 = L^4T^{-4}$, and $1/Kc = L^{-1}T^2$, whence $1/k^4Kc = L^3T^{-2}$, the dimensions of h. This expression makes h depend upon accurately known constants, K and c. The numerical value of k is unity, and

$$1/Kc = 1/3.290\times10^{15}\times3\times10^{10} = 10.13171\times10^{-27} \quad \ldots\ldots \quad (4)$$

Also
$$8^5/15^4 = 32768/50625 = 0.647269. \quad \ldots \quad (5)$$

Multiplying these together, we obtain as the numerical value of h,
$$h = 6.5579 \times 10^{-27}. \quad \ldots\ldots\ldots \quad (6)$$

The velocity of light has been taken as the even number 3×10^{10}; the Rydberg constant as the even number 3.29×10^{15}, and the decimal places have been retained on this account. When the best values of these two constants are employed the velocity of light will be a very little smaller, making h a very little larger. It may be remarked that the value just obtained is very close to the value of h, namely 6.56×10^{-27}, which was obtained by Millikan as the best figure representing the total result of his experiments on the emission of electrons from fresh metallic surfaces in vacuo by light of different frequencies. This was his machine-shop-in-vacuo experiment, carried out primarily to verify the Einstein equation that makes energy proportional to frequency. This experiment resulted in an unusually good straight

line, all the observed points lying close to the line throughout the whole range of frequencies observed, and this established the proportionality in a very satisfactory manner. The slope of the line gave the ratio of e to b, whence b was calculated from the previous knowledge of e.

And again, by means of the Lorentz mass formula, given in (160) above, namely

$$a_H = \frac{4}{5m_H k}\left(\frac{2e}{c}\right)^2, \quad \ldots \ldots \ldots \quad (7)$$

we may eliminate a_H from equation (1) and find

$$b = \left(\frac{16e^2}{15m_H kc^2}\right)^4 \frac{8K^3}{c}. \quad \ldots \ldots \quad (8)$$

And by means of the expression for $2K$ in (149), namely

$$2K = m_H\left(\frac{c}{e}\right)^2, \quad \ldots \ldots \ldots \ldots \quad (9)$$

we find another expression for b:

$$b = \left(\frac{16}{15k}\right)^4 \frac{e^2}{m_H c^3}. \quad \ldots \ldots \quad (10)$$

The dimensions of this are correct, for the denominator is dimensionless, since $m_H c^3$ is the fourth power of a velocity and k^4 is the reciprocal of the fourth power of a velocity. This leaves the dimensions of b the same as those of e^2, as it should be according to the space-time system, as pointed out above.

Numerically, if we use for e^2/m_H the value 1.36778×10^5, obtained in (154) above, and take $c = 3 \times 10^{10}$, we find

$$b = \left(\frac{16}{15k}\right)^4 \times \frac{1.36778 \times 10^5}{27 \times 10^{30}} = \frac{1}{k^4} \times 1.294538$$
$$\times 5.06585 \times 10^{-27} = 6.5579 \times 10^{-27}, \quad \ldots \quad (11)$$

the same value found in (6).

Referring to the original equation (1), it is to be observed that the specific inductive capacity, k, as well as the masses of the electron and nucleus, are absent, so that no uncertainty enters this expression because of any possible doubt that specific inductive capacity is the reciprocal of a velocity, and that mass is a velocity. The quantity $2K$ appears as representing the frequency of revolution of the electrons in the normal hydrogen atom, and as possibly that of the nucleus itself. There is no immediately assignable reason why the $\frac{1}{3}$d part of the radius should appear in the expression instead of the whole radius; but it is not unnatural that $\frac{1}{3}$d should be required in connection with a spherical shape. The volume of a conical portion of a sphere is $\frac{1}{3}$d the radius times the area included by the base.

It seems as if many of these incomprehensible matters might become more comprehensible if we did not have to use such large units of length and of time as the centimeter and the second in measuring atomic quantities, which are so small in comparison. Accordingly, let us adopt as the unit of length the distance that light travels while the electrons are making just one revolution in the hydrogen atom, namely in a time $1/2K$ seconds. And let us take the time of one revolution as a unit of time instead of using the second.

To convert the several kinds of quantities that continually occur over into this new system, we have

A unit of length (new) $= 3 \times 10^{10}/2K = 4.559{,}27 \times 10^{-6}$ cm.
A unit of time (new) $= 1/2K = .151{,}975 \times 10^{-15}$ seconds.
One second (old) $= 2K = 6.58 \times 10^{15}$ new units.

To obtain the new values of other quantities, involving powers of L or T or both in the dimensional formula, multiply each L by $2K/3 \times 10^{10}$, and each T by $2K$. For example, the velocity of light becomes

$$c = 3 \times 10^{10} \text{ cm. per sec. (old)} = 3 \times 10^{10} \frac{2K}{3 \times 10^{10}} \frac{1}{2K}$$
$$= 1 \text{ unit} = LT^{-1},$$

and is unity on the new system of units, as is evident because light travels by hypothesis a unit distance in the time of one revolution, or unit time. So any velocity on the new system is numerically equal to that on the old divided by 3×10^{10}.

Mass has the dimensions of a velocity. Hence one gram on the old system becomes $0.333,333 \times 10^{-10}$ units on the new system, and the unit of mass on the new system is 3×10^{10} grams, which would be represented by a cube of water about 31.07 meters on an edge. This unit of mass is thus inconveniently large for a practical system of units. The mass of the hydrogen atom becomes

$$m_H = 1.658 \times 10^{-24}/3 \times 10^{10}$$
$$= 0.55267 \times 10^{-34} \text{ units} = LT^{-1}.$$

And the mass of the electron becomes

$$m_0 = .898 \times 10^{-27}/3 \times 10^{10} = 0.29933 \times 10^{-37} \text{ units.}$$

Specific inductive capacity becomes

$$k = 1 \text{ (old system)} = L^{-1}T = 3 \times 10^{10},$$

and is numerically equal to the velocity of light on the old system.

Twice the Rydberg constant becomes

$$2K = T^{-1} = 6.58 \times 10^{15} \text{ (old)}$$

$$= 6.58 \times 10^{15} \frac{1}{2K}$$

$$= \text{unity in the new system.}$$

The equation (149) for the Rydberg constant is

$$2K = m_H\left(\frac{c}{e}\right)^2,$$

and in this, since both $2K$ and c are numerically equal to unity, it appears that m_H is numerically equal to e^2, although not having the same dimensions. Hence on the new system

$e^2 = 0.55267 \times 10^{-34}$, and $e = 0.743 \times 10^{-17}$ units.

The equation (149), when written

$$\text{Energy} = 2Ke^2 = m_Hc^2 = L^3T^{-3},$$

represents energy, namely the probable energy content of the hydrogen nucleus given above (see page 131) as 1.492×10^{-3} ergs. Since, however, $2K$ and c are now unity, the mass of the nucleus is numerically equal to the energy content of the nucleus when expressed in the new unit of energy. The new unit of energy becomes

$$\text{One new unit of energy} = 27 \times 10^{30} \text{ ergs},$$

and $\quad$ One erg $= 3.703,703 \times 10^{-28}$ new units.

Since one joule is equal to 10^7 ergs, it is equivalent to 3.7037×10^{-21} new units of energy. The number of nuclei in one gram of hydrogen is equal to the Avogadro constant, 6.062×10^{23}. The total energy content of one gram of hydrogen is then

$$\begin{aligned} 6.062 \times 10^{23} \times 1.492 \times 10^{-3} &= 9.0457 \times 10^{20} \text{ ergs} \\ &= 9.0457 \times 10^{13} \text{ joules.} \end{aligned}$$

If it is now supposed that this energy may by some means be extracted at the rate of 1000 watts, or 1 kilowatt continuously, the total supply in one gram of hydrogen will last at this rate for

$$9.0457 \times 10^{13}/1000 = 9.0457 \times 10^{10} \text{ seconds.}$$

This number of seconds is about equivalent to 2870 years' time. The practical unit has been used in this calculation as being more familiar. We have a better appreciation of the enormous amount of energy stored in the gram of hydrogen when expressed in units easily grasped.

In conclusion it seems worth remarking that Planck's constant, h, takes the very simple numerical value

$$h = \left(\frac{16}{15k}\right)^4 = \left(\frac{16}{15 \times 3 \times 10^{10}}\right)^4 = (.3\dot{5}\dot{5})^4 \times 10^{-40}$$
$$= 1.598 \times 10^{-42},$$

on the new system of units of length and time because of the relation in (10) above, in which the factor $e^2/m_H c^3$ becomes numerically equal to unity because $c = 1$, and $e^2 = m_H$ numerically. k is numerically equal to the velocity of light on the C.G.S. system. The dimensions of the numerical expression for h just given are not complete without the factor as in (10). This case is very much like the common practice in the present system of units of suppressing the specific inductive capacity, which alters the dimensions of the expressions. If we had always used these new units instead of the centimeter and the second, this factor $e^2/m_H c^3$ might have been customarily suppressed, and we should have a false idea of the dimensions of h.

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